Učebný text (v anglickom jazyku)

P R U Ž I N O V Ý A P R Ú T O V Ý P R V O K

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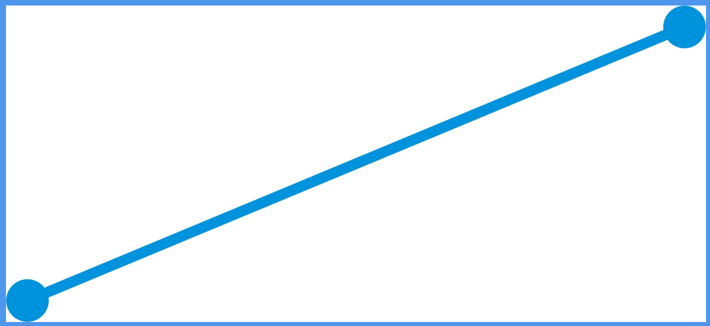
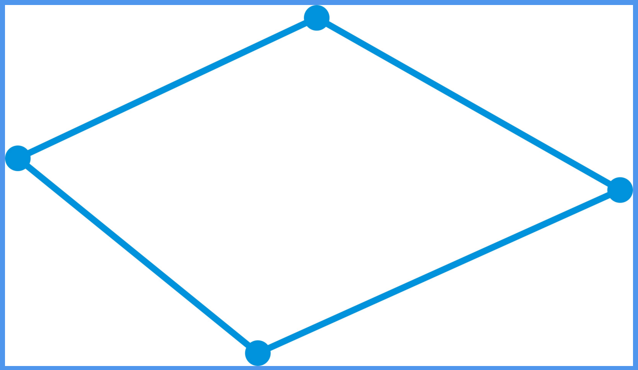
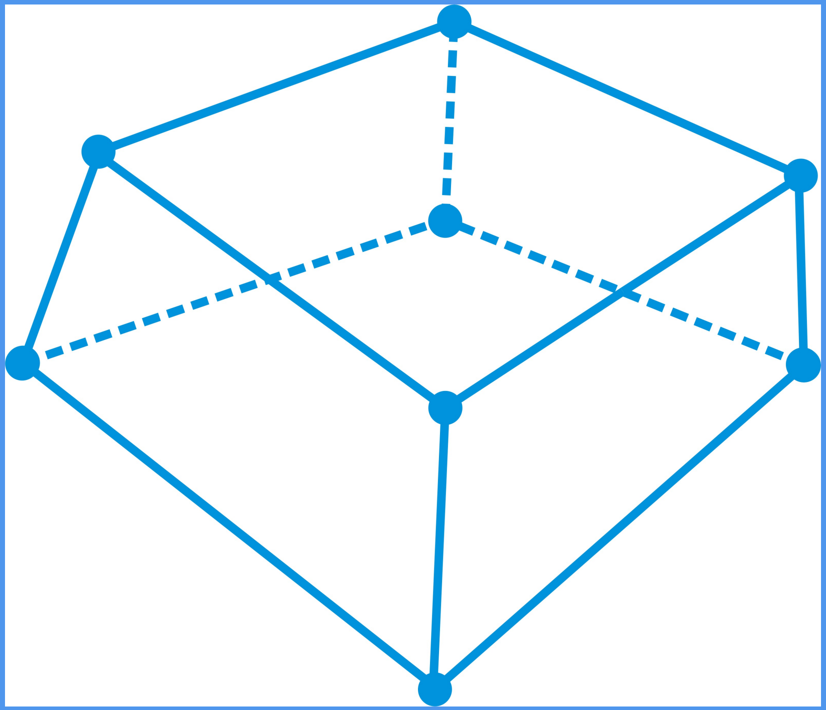
S podporou projektu KEGA 060STU-4/2016

Types of Finite Elements

1D (Line) Elements	2D (Plane) Elements
	
(Spring, truss, beam, pipe, etc.)	(Membrane, plate, shell, etc.)

3-D (Solid) Element

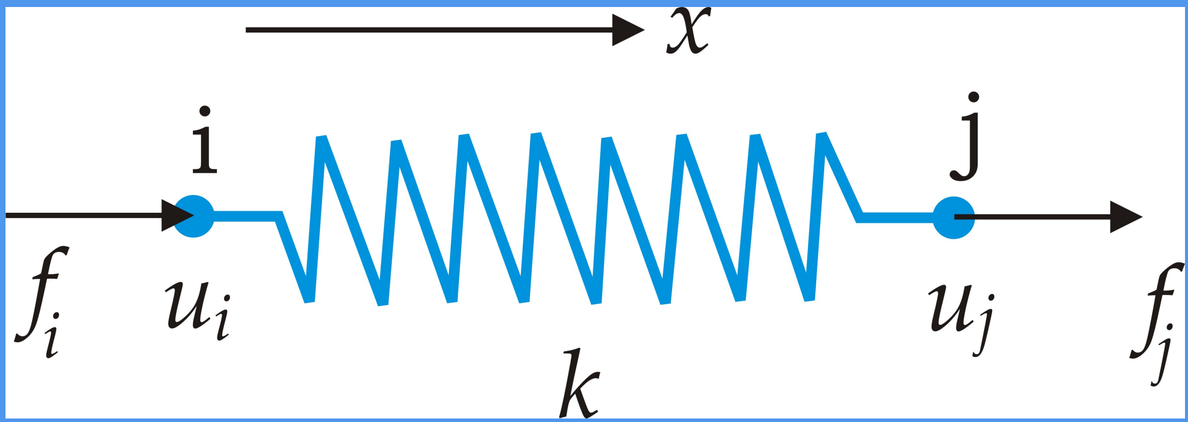


(3-D fields - temperature, displacement, stress, flow velocity)

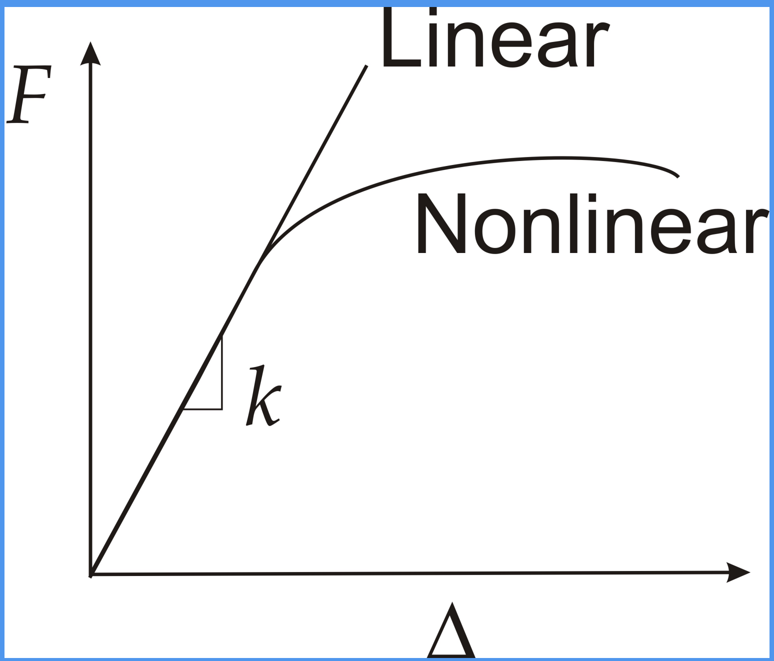
S P R I N G E L E M E N T

”Everything important is simple.”

One Spring Element



Two nodes: $i,\ j$
Nodal displacements: $u_i,\ u_j$
Nodal forces: $f_i,\ f_j$
Spring constant (stiffness): k



Spring force displacement relationships:

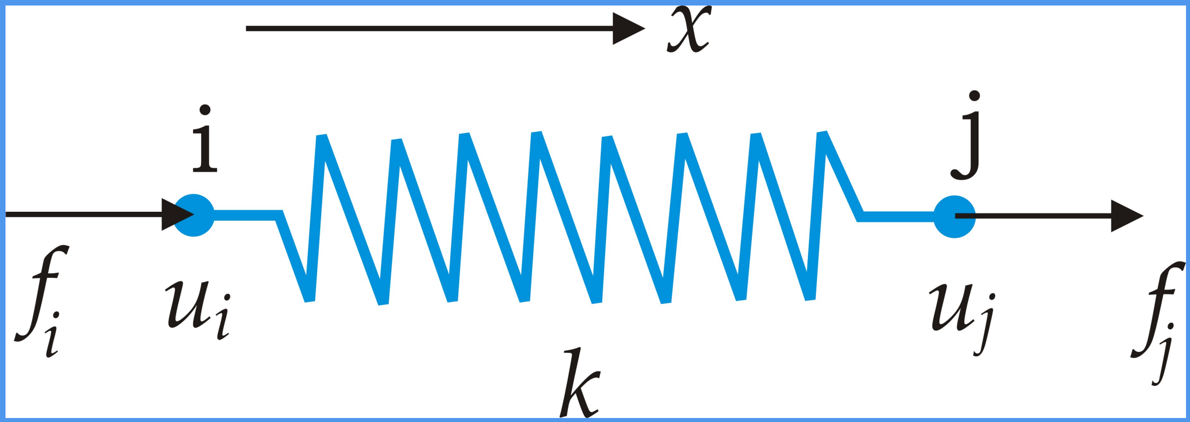
$$F = k\,\Delta$$

with

$$\Delta = u_j - u_i$$

$k = F\,/\,\Delta\ \ (> 0)$ is the force needed to produce a unit stretch.
We only consider ***linear problems*** in this lecture.

One Spring Element



Consider the equilibrium of forces for the spring. At node i, we have

$$f_i = -F = k(u_j - u_i) = ku_i - ku_j$$

and at node j,

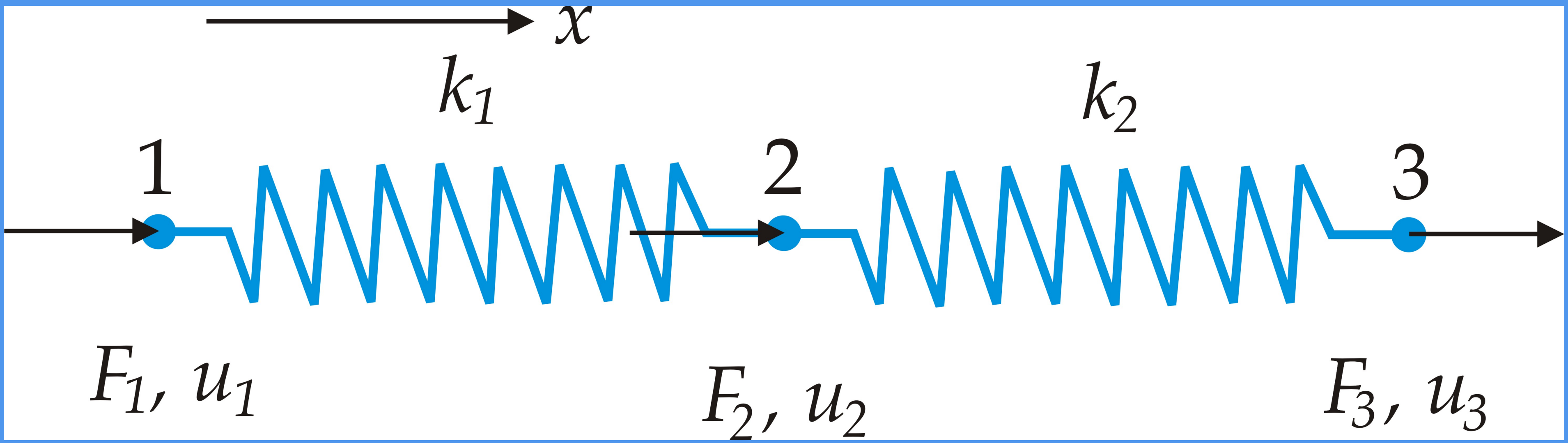
$$f_j = F = k(u_j - u_i) = -ku_i + ku_j$$

In matrix form,

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{Bmatrix} f_i \\ f_j \end{Bmatrix}, \qquad \text{or,} \qquad \mathbf{k}\,\mathbf{u} = \mathbf{f},$$

where $\mathbf{k}$ = (element) stiffness matrix, $\mathbf{u}$ = (element nodal) displacement vector, $\mathbf{f}$ = (element nodal) force vector.

Spring System



For element 1,

$$\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1^1 \\ f_2^1 \end{Bmatrix}$$

element 2,

$$\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_1^2 \\ f_2^2 \end{Bmatrix}$$

where f_i^m is the (internal) force acting on *local* node i of element m ($i = 1, 2$).

Assemble the stiffness matrix for the whole system:

Consider the equilibrium of forces at node 1,

$$F_1 = f_1^1$$

at node 2,

$$F_2 = f_2^1 + f_1^2$$

and node 3,

$$F_3 = f_2^2.$$

That is,

$$\begin{aligned} F_1 &= k_1 u_1 - k_1 u_2 \\ F_2 &= -k_1 u_1 + (k_1 + k_2) u_2 - k_2 u_3 \\ F_3 &= -k_2 u_2 + k_2 u_3 \end{aligned}$$

In matrix form,

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} \qquad or \qquad \mathbf{K}\,\mathbf{U} = \mathbf{F}$$

$\mathbf{K}$ is the stiffness matrix (structure matrix) for the spring system.

An alternative way of assembling the whole stiffness matrix:

”Enlarging” the stiffness matrices for element 1 and 2, we have

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_1^1 \\ f_2^1 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f_1^2 \\ f_2^2 \end{Bmatrix}$$

Adding the two matrix equations (*superposition*), we have

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ 0 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_1^1 \\ f_2^1 + f_1^2 \\ f_2^2 \end{Bmatrix}.$$

This is the same equation we derived by using the force equilibrium concept.

Boundary and load conditions:

Assuming, $u_1 = 0$ and $F_2 = F_3 = P$

we have

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ P \\ P \end{Bmatrix}$$

which reduces to

$$\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} P \\ P \end{Bmatrix}$$

and

$$F_1 = -k_1 u_2$$

Unknowns are

$$\mathbf{U} = \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$$ and the reaction F_1 (if desired).

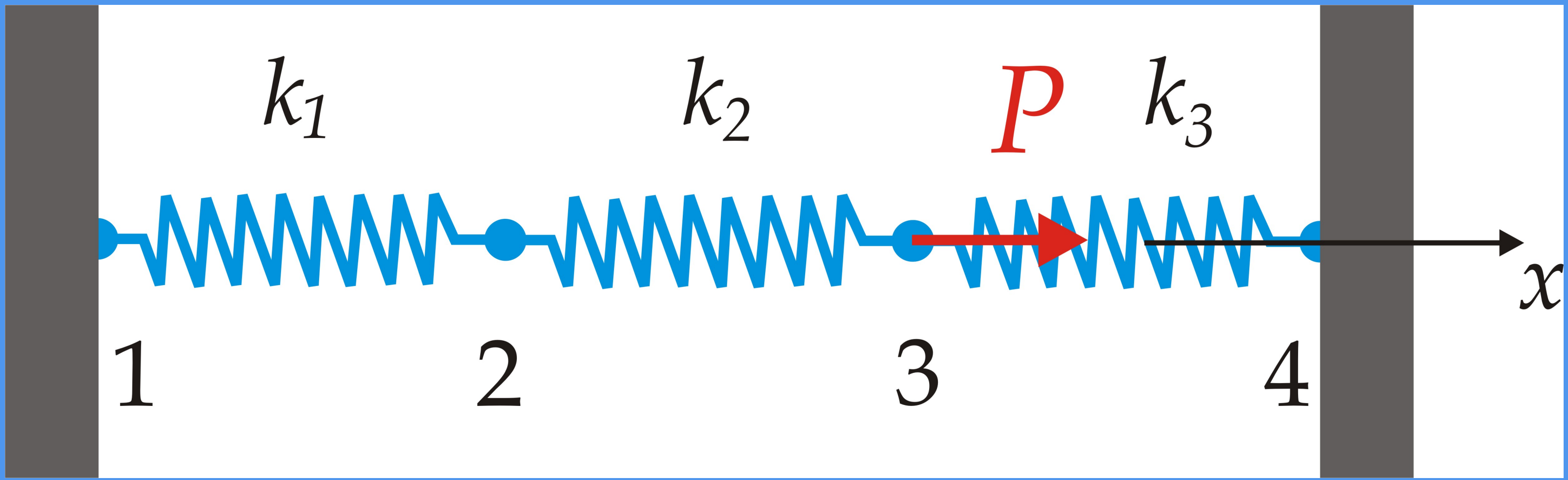
Solving the equations, we obtain the displacements

$$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 2P / k_1 \\ 2P / k_1 + P / k_2 \end{Bmatrix}$$

and the reaction force

$$F_1 = -2P.$$

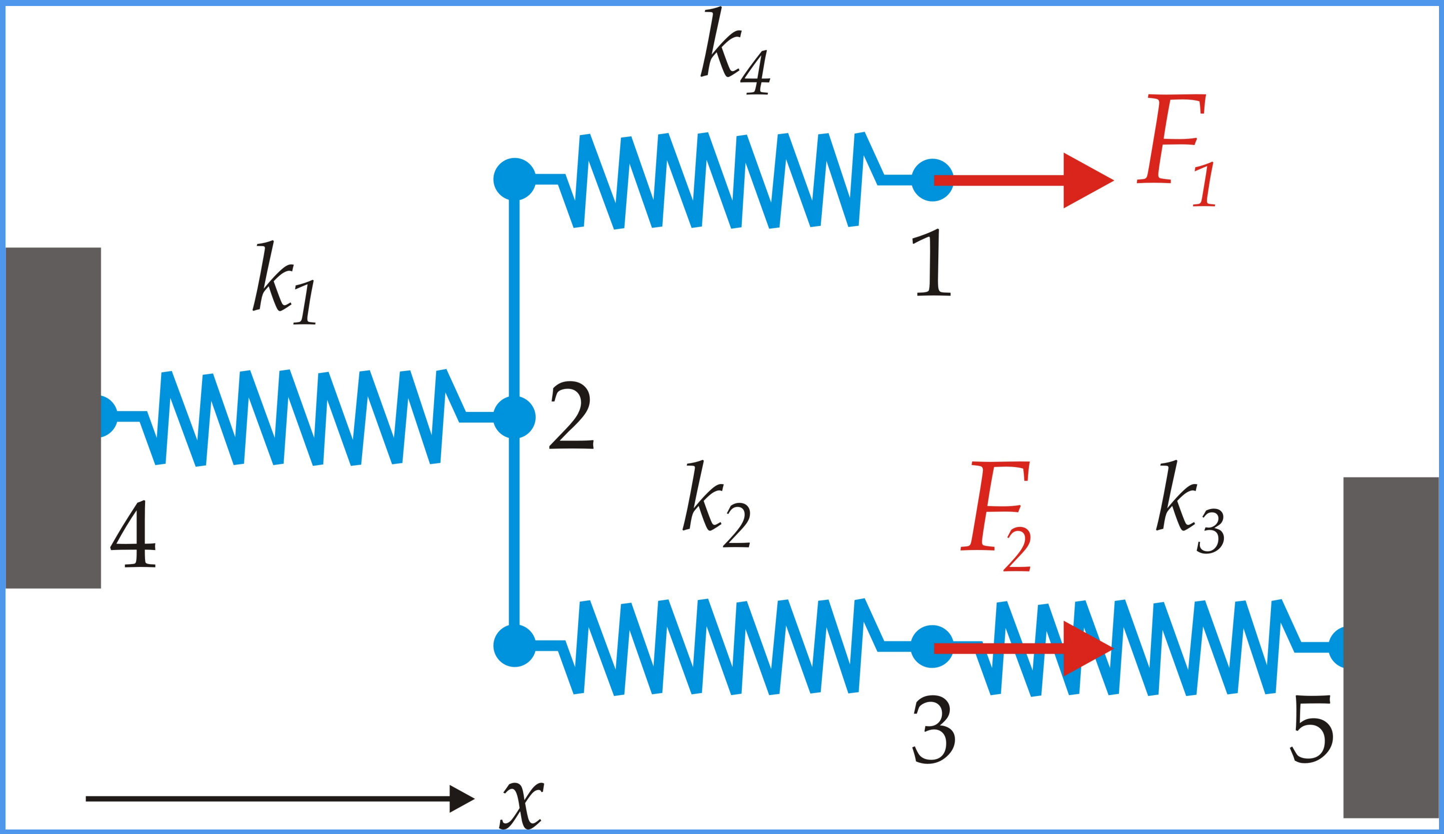
Example 1:



Given: For the spring system show above,
$k_1 = 100$ N/mm, $k_2 = 200$ N/mm, $k_3 = 100$ N/mm
$P = 500$ N, $u_1 = u_4 = 0$

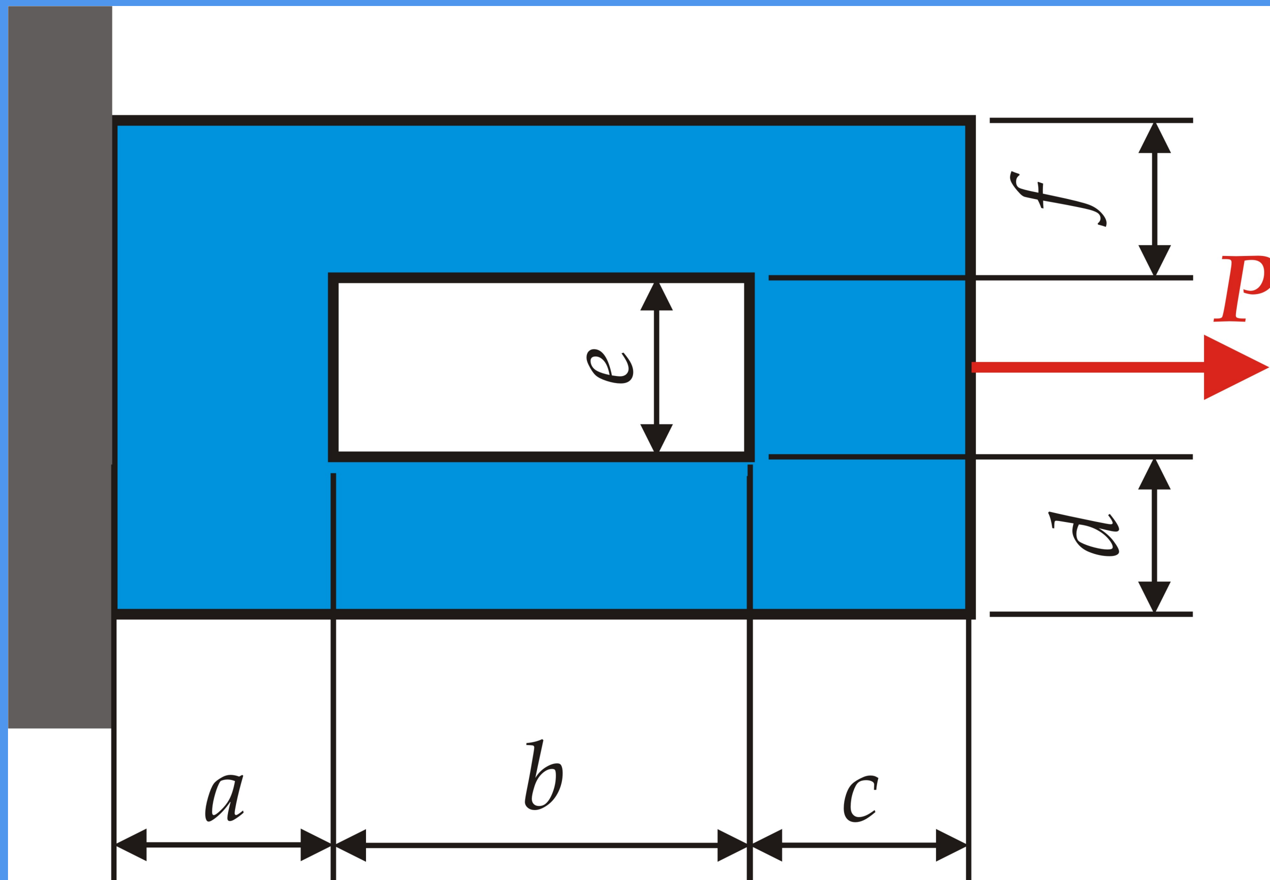
Find: (a) the global stiffness matrix
(b) displacements of nodes 2 and 3
(c) the reaction forces at nodes 1 and 4
(d) the force in the spring 2

Example 2:



Problem: For the spring system with arbitrarily numbered nodes and elements, as shown above, find the global stiffness matrix.

Example 3:



Problem: A steel plate is subjected to an axial load, as shown in Figure. Approximate the deflection and average stresses along the plate. The plate is 2 mm in thick and has Young's modulus $E = 2.1\,10^6$ MPa.

BAR AND TRUSS ELEMENT

Linear Static Analysis

Most structural analysis problems can be treated as *linear static* problems, based on the following assumptions

1. *Small deformations* (loading pattern is not changed duo to the deformed shape)

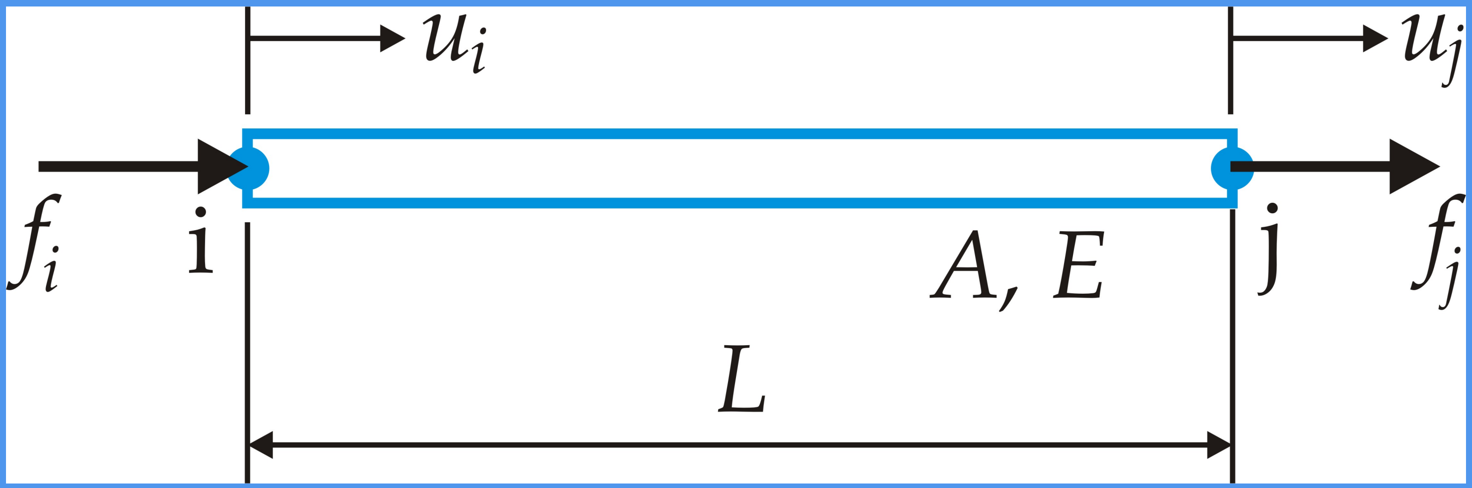
2. *Elastic materials* (no plasticity or failures)

3. *Static loads* (the load is applied to the structure in a slow or steady fashion)

Linear analysis can provide most of the information about the behavior of structure, and can be a good approximation for many analysis. It is also the bases of nonlinear analysis in most of the cases.

Bar Element (1-D)

Consider a uniform prismatic bar:



L	length
A	cross-section area
E	Young's modulus (elastic modulus)
$u = u(x)$	displacement
$\varepsilon = \varepsilon(x)$	strain
$\sigma = \sigma(x)$	stress

Strain-displacement relation:

$$\varepsilon = \frac{\mathrm{d}u}{\mathrm{d}x}$$

Stress-strain relation:

$$\sigma = E\varepsilon$$

Stiffnes Matrix — Direct Method

Assuming that the displacement u is *varying linearly* along the axis of the bar, i.e.,

$$u(x) = \left(1 - \frac{x}{L}\right) u_i + \frac{x}{L} u_j$$

we have

$$\varepsilon = \frac{u_j - u_i}{L} = \frac{\Delta}{L} \qquad (\Delta = \text{elongation})$$

$$\sigma = E\varepsilon = \frac{E\Delta}{L}$$

We also have

$$\sigma = \frac{F}{A} \qquad (F = \text{force in bar})$$

Thus, $\sigma = E\varepsilon = \dfrac{E\Delta}{L}$ and $\sigma = \dfrac{F}{A}$ lead to

$$F = \frac{EA}{L}\Delta = k\Delta$$

where $k = \dfrac{EA}{L}$ is the stiffness of the bar.

Stiffnes Matrix — Direct Method

The *bar is acting like a spring* in this case and we conclude that element stiffness matrix is

$$\mathbf{k} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} = \begin{bmatrix} \dfrac{EA}{L} & -\dfrac{EA}{L} \\ -\dfrac{EA}{L} & \dfrac{EA}{L} \end{bmatrix} \quad \text{or} \quad \mathbf{k} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

This can be verified by considering the equilibrium of the forces at the two nodes.

Element equilibrium equation is

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_i \\ f_j \end{Bmatrix}$$

Degree of Freedom (dof)

Number of components of the displacement vector at a node.

For 1-D bar element: one dof at each node.

Stiffnes Matrix — Direct Method

Physical Meaning of the Coefficients in $\mathbf{k}$

The jth column of $\mathbf{k}$ (there $j = 1$ or 2) represents the forces applied to the bar to maintain a deformed shape with unit displacement at node j and zero displacement at the other node.

Stiffnes Matrix — A Formal Method

We derive the same stiffness matrix for bar using a formal approach which can be applied to many other more complicated situations.
Define two *linear shape functions* as follows

$$N_i(\xi) = 1 - \xi, \qquad N_j(\xi) = \xi\,,$$

where

$$\xi = \frac{x}{L}, \qquad 0 \leq \xi \leq 1$$

From $u(x) = \left(1 - \frac{x}{L}\right) u_i + \frac{x}{L} u_j$ we can write the displacement as

$$u(x) = u(\xi) = N_i(\xi) u_i + N_j(\xi) u_j$$

or

$$u = \begin{bmatrix} N_i & N_j \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \mathbf{N}\mathbf{u}. \qquad (1)$$

Strain is given by $\varepsilon = \frac{\mathrm{d}u}{\mathrm{d}x}$ and (1) as

$$\varepsilon = \frac{\mathrm{d}u}{\mathrm{d}x} = \left[\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{N}\right]\mathbf{u} = \mathbf{B}\mathbf{u}$$

Stiffnes Matrix — A Formal Method

where $\mathbf{B}$ is the element *strain-displacement matrix*, which is

$$\mathbf{B} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\begin{array}{cc} N_i(\xi) & N_j(\xi) \end{array}\right] = \frac{\mathrm{d}}{\mathrm{d}\xi}\left[\begin{array}{cc} N_i(\xi) & N_j(\xi) \end{array}\right] \bullet \frac{\mathrm{d}\xi}{\mathrm{d}x}$$

i.e., $$\mathbf{B} = \left[\begin{array}{cc} -\frac{1}{L} & \frac{1}{L} \end{array}\right]$$

Stress can be written as

$$\sigma = E\varepsilon = E\mathbf{Bu}$$

Consider the *strain energy* stored in the bar

$$U = \frac{1}{2}\int_V \sigma^T \varepsilon \mathrm{d}V = \frac{1}{2}\int_V \left(\mathbf{u}^T\mathbf{B}^T E\mathbf{Bu}\right)\mathrm{d}V = \frac{1}{2}\mathbf{u}^T\left[\int_V \left(\mathbf{B}^T E\mathbf{B}\right)\mathrm{d}V\right]\mathbf{u}$$

where $\varepsilon = \frac{\mathrm{d}u}{\mathrm{d}x} = \left[\frac{\mathrm{d}}{\mathrm{d}x}\mathbf{N}\right]\mathbf{u} = \mathbf{Bu}$ and $\sigma = E\varepsilon = E\mathbf{Bu}$ have been used.

The work done by the two nodal forces is

$$W = \frac{1}{2}f_i u_i + \frac{1}{2}f_j u_j = \frac{1}{2}\mathbf{u}^T\mathbf{f}$$

Stiffnes Matrix — A Formal Method

For conservative system, we state that

$$U = W$$

which gives

$$\frac{1}{2}\mathbf{u}^T \left[\int_V \left(\mathbf{B}^T E \mathbf{B}\right) \mathrm{d}V\right] \mathbf{u} = \frac{1}{2}\mathbf{u}^T \mathbf{f}$$

We can conclude that

$$\left[\int_V \left(\mathbf{B}^T E \mathbf{B}\right) \mathrm{d}V\right] \mathbf{u} = \mathbf{f} \qquad \text{or} \qquad \mathbf{k}\,\mathbf{u} = \mathbf{f}$$

where

$$\mathbf{k} = \int_V \left(\mathbf{B}^T E \mathbf{B}\right) \mathrm{d}V \qquad (2)$$

is the *element stiffness matrix*.

Stiffnes Matrix — A Formal Method

Expression (2) is general result which can be used for construction of other types of elements. This expression can also be derived using other more rigorous approaches, such as the *Principle of Minimum Potentional Energy*, or the *Galerkin's Methods*.

Now, we evaluate (2) for the bar element by using $\mathbf{B} = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}$

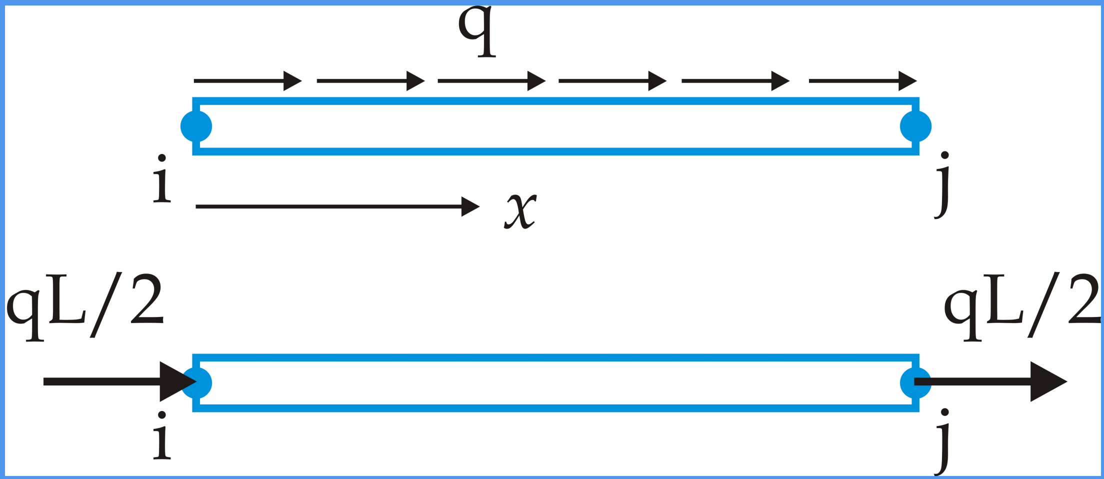
$$\mathbf{k} = \int_0^L \begin{Bmatrix} -1/L \\ 1/L \end{Bmatrix} E \begin{bmatrix} -1/L & 1/L \end{bmatrix} A \mathrm{d}x = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

which is the same as we derived using the direct method.

Note that from and , the strain energy in the element can be written as

$$U = \frac{1}{2} \mathbf{u}^T \mathbf{k} \mathbf{u}$$

Distributed load



Uniformly distributed axial load q (N/mm, N/m, lb/in) can be converted to two equivalent nodal forces of magnitude $qL/2$.
We verify this by considering the work done by the load q,

$$W_q = \int_0^L \frac{1}{2} uq\,\mathrm{d}x = \frac{1}{2}\int_0^1 u(\xi)q(L\mathrm{d}\xi) = \frac{qL}{2}\int_0^1 u(\xi)\mathrm{d}\xi =$$

$$= \frac{qL}{2}\int_0^1 \begin{bmatrix} N_i(\xi) & N_j(\xi) \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} \mathrm{d}\xi = \frac{qL}{2}\int_0^1 \begin{bmatrix} 1-\xi & \xi \end{bmatrix} \mathrm{d}\xi \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} =$$

$$= \frac{1}{2}\begin{bmatrix} \frac{qL}{2} & \frac{qL}{2} \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \frac{1}{2}\begin{bmatrix} u_i & u_j \end{bmatrix} \begin{Bmatrix} qL/2 \\ qL/2 \end{Bmatrix}$$

Distributed load

that is,

$$W_q = \frac{1}{2}\mathbf{u}^T\mathbf{f}_q \text{with}\,\mathbf{f}_q = \begin{Bmatrix} qL/2 \\ qL/2 \end{Bmatrix}$$

Thus, from the $U = W$ concept for the element, we have

$$\frac{1}{2}\mathbf{u}^T\mathbf{k}\mathbf{u} = \frac{1}{2}\mathbf{u}^T\mathbf{f} + \frac{1}{2}\mathbf{u}^T\mathbf{f}_q$$

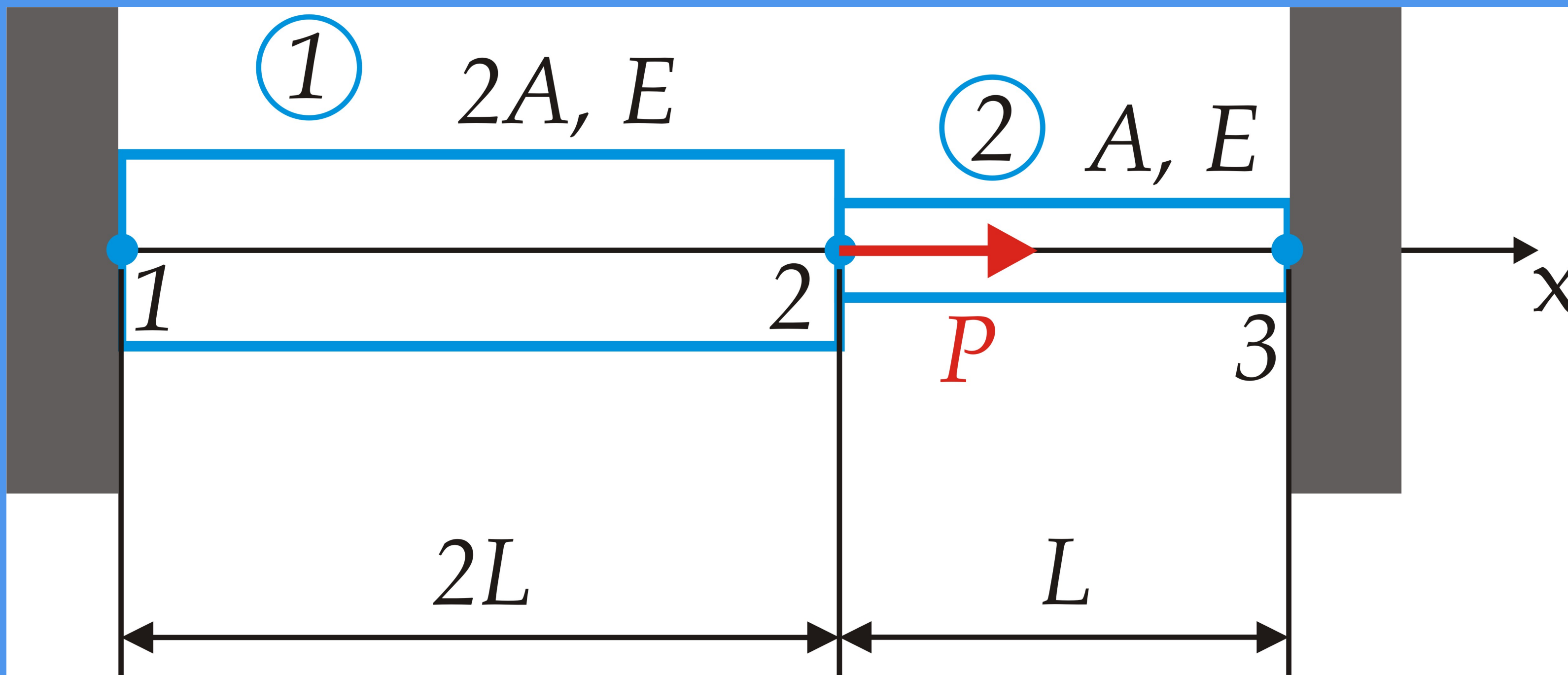
which yields

$$\mathbf{k}\mathbf{u} = \mathbf{f} + \mathbf{f}_q.$$

The new nodal force vector is

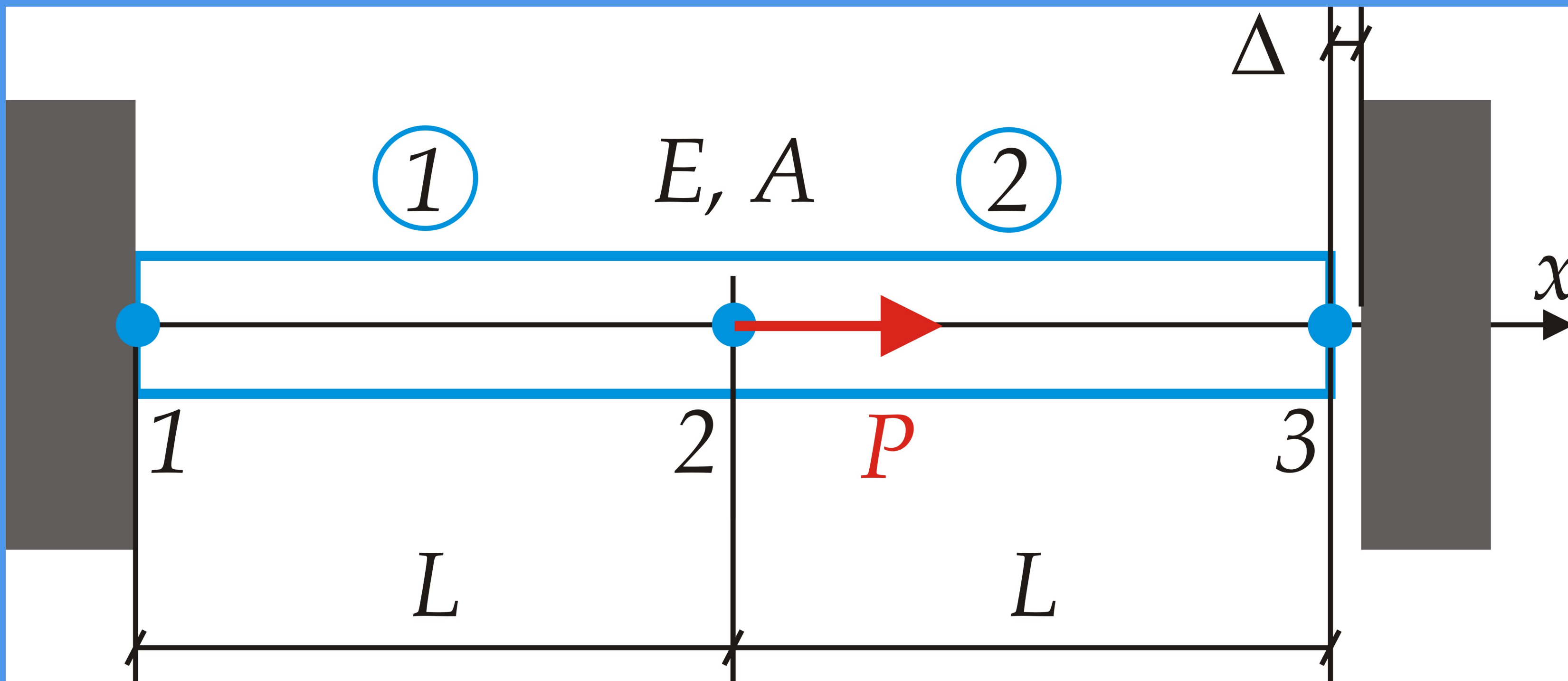
$$\mathbf{f} + \mathbf{f}_q = \begin{Bmatrix} f_i + qL/2 \\ f_j + qL/2 \end{Bmatrix}$$

Example 4:



Problem: Find the stresses in two bar assembly which is loaded with force P, and constrained at the two ends, as shown in the figure.

Example 5:



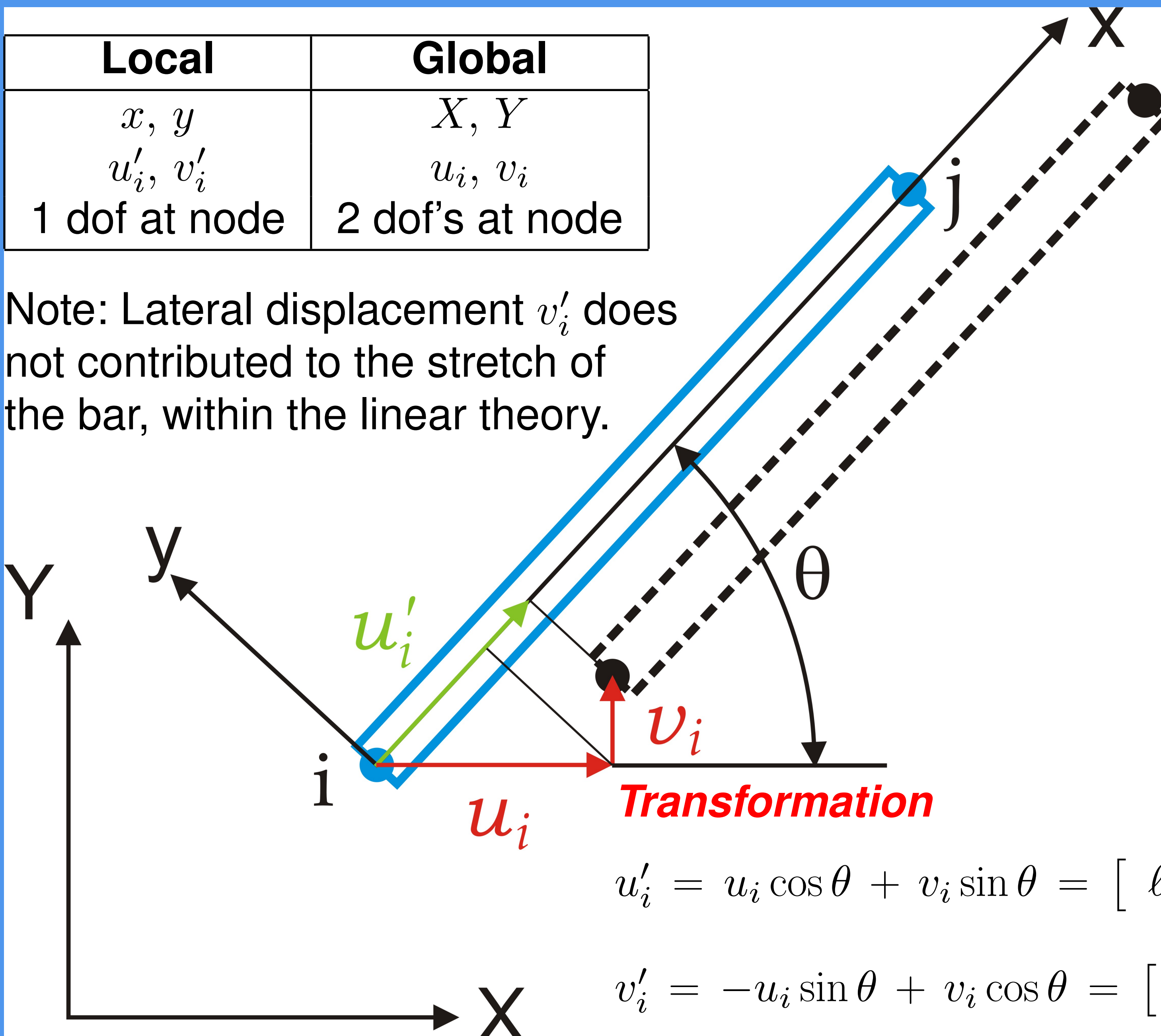
Problem: Determine the support reaction forces at the two ends of the bar shown above, given the following,

$$P = 6.0 \times 10^4 \text{ N}, \quad E = 2.0 \times 10^4 \text{ N/mm}^2,$$
$$a = 250\,\text{mm}^2, \quad L = 150\,\text{mm}, \quad \Delta = 1.2\,\text{mm}$$

Bar Elements in 2-D Space

Local	Global
$x,\ y$	$X,\ Y$
$u'_i,\ v'_i$	$u_i,\ v_i$
1 dof at node	2 dof's at node

Note: Lateral displacement v'_i does not contributed to the stretch of the bar, within the linear theory.



Transformation

$$u'_i = u_i \cos\theta + v_i \sin\theta = \begin{bmatrix} \ell & m \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \end{Bmatrix}$$

$$v'_i = -u_i \sin\theta + v_i \cos\theta = \begin{bmatrix} -m & \ell \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \end{Bmatrix}$$

where $\ell = \cos\theta,\ m = \sin\theta$

Bar Elements in 2-D Space

In matrix form,

$$\left\{ \begin{array}{c} u_i' \\ v_i' \end{array} \right\} = \left[\begin{array}{cc} \ell & m \\ -m & \ell \end{array} \right] \left\{ \begin{array}{c} u_i \\ v_i \end{array} \right\} \qquad \text{or,} \qquad \mathbf{u}_i' = \tilde{\mathbf{T}} \mathbf{u}_i$$

where the *transformation matrix*

$$\tilde{\mathbf{T}} = \left[\begin{array}{cc} \ell & m \\ -m & \ell \end{array} \right]$$

is *orthogonal*, that is, $\tilde{\mathbf{T}}^{-1} = \tilde{\mathbf{T}}^T$.
For the two nodes of the bar element, we have

$$\left\{ \begin{array}{c} u_i' \\ v_i' \\ u_j' \\ v_j' \end{array} \right\} = \left[\begin{array}{cccc} \ell & m & 0 & 0 \\ -m & \ell & 0 & 0 \\ 0 & 0 & \ell & m \\ 0 & 0 & -m & \ell \end{array} \right] \left\{ \begin{array}{c} u_i \\ v_i \\ u_j \\ v_j \end{array} \right\} \text{or, } \mathbf{u}' = \mathbf{T}\mathbf{u} \qquad \text{with } \mathbf{T} = \left[\begin{array}{cc} \tilde{\mathbf{T}} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathbf{T}} \end{array} \right]$$

The nodal forces are transformed in the same way,

$$\mathbf{f}' = \mathbf{T}\mathbf{f}.$$

Stiffness Matrix in 2-D Space

In the local coordinate system, we have

$$\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_i' \\ u_j' \end{Bmatrix} = \begin{Bmatrix} f_i' \\ f_j' \end{Bmatrix}$$

Augmenting this equation, we write

$$\frac{EA}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u_i' \\ v_i' \\ u_j' \\ v_j' \end{Bmatrix} = \begin{Bmatrix} f_i' \\ 0 \\ f_j' \\ 0 \end{Bmatrix} \qquad \text{or,} \qquad \mathbf{k}'\mathbf{u}' = \mathbf{f}'$$

Using transformations given in $\mathbf{u}' = \mathbf{T}\mathbf{u}$ and $\mathbf{f}' = \mathbf{T}\mathbf{f}$, we obtain

$$\mathbf{k}'\mathbf{T}\mathbf{u} = \mathbf{T}\mathbf{f}$$

Multiplying both sides by $\mathbf{T}^T$ and noticing that $\mathbf{T}^T\mathbf{T} = \mathbf{I}$, we obtain

$$\mathbf{T}^T\mathbf{k}'\mathbf{T}\mathbf{u} = \mathbf{f}$$

Stiffness Matrix in 2-D Space

Thus, the element stiffness matrix $\mathbf{k}$ in the global coordinate system is

$$\mathbf{k} = \mathbf{T}^T \mathbf{k}' \mathbf{T}$$

which is a 4×4 symmetric matrix.
Explicit form,

$$\mathbf{k} = \frac{EA}{L} \begin{bmatrix} \ell^2 & \ell m & -\ell^2 & -\ell m \\ \ell m & m^2 & -\ell m & -m^2 \\ -\ell^2 & -\ell m & \ell^2 & \ell m \\ -\ell m & -m^2 & \ell m & m^2 \end{bmatrix}.$$

Calculation of the ***directional cosines*** ℓ and m:

$$\ell = \cos\theta = \frac{X_j - X_i}{L}, \qquad m = \sin\theta = \frac{Y_j - Y_i}{L}.$$

The structure stiffness matrix is assembled by using the element stiffness matrices in the usual way as in the 1-D case.

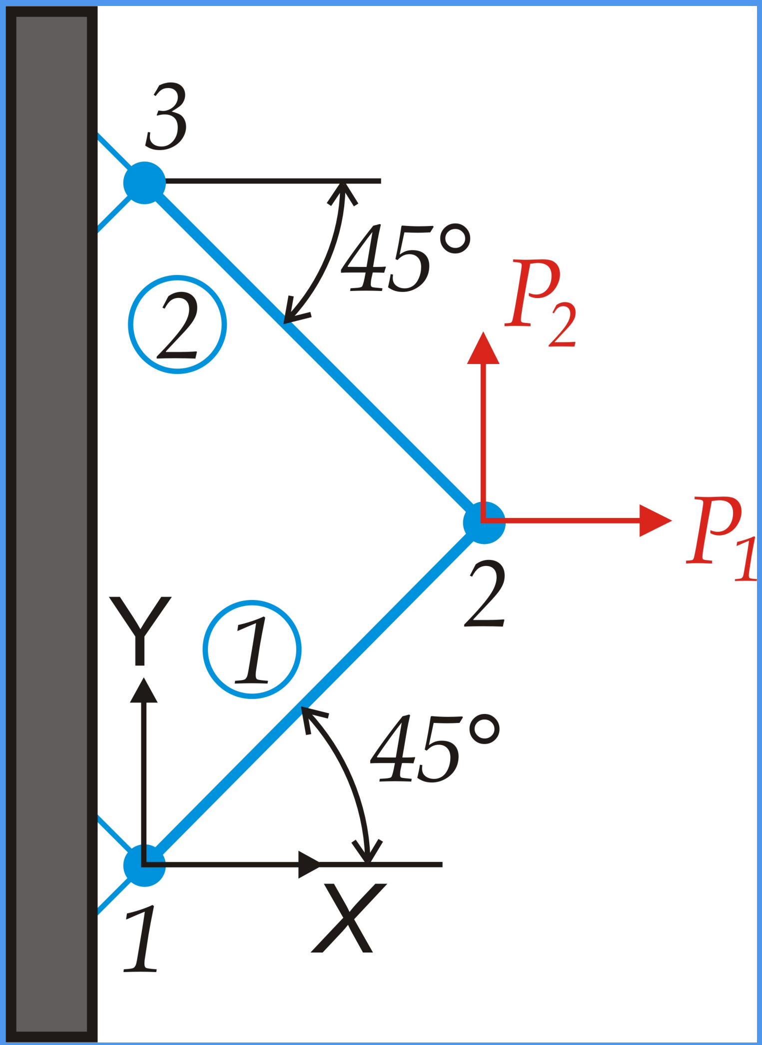
Element Stress

$$\sigma = E\varepsilon = E\mathbf{E}\begin{Bmatrix} u_i' \\ u_j' \end{Bmatrix} = E\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}\begin{bmatrix} \ell & m & 0 & 0 \\ 0 & 0 & \ell & m \end{bmatrix}\begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix}$$

That is,

$$\sigma = \frac{E}{L}\begin{bmatrix} -\ell & -m & \ell & m \end{bmatrix}\begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \end{Bmatrix}$$

Example 6:

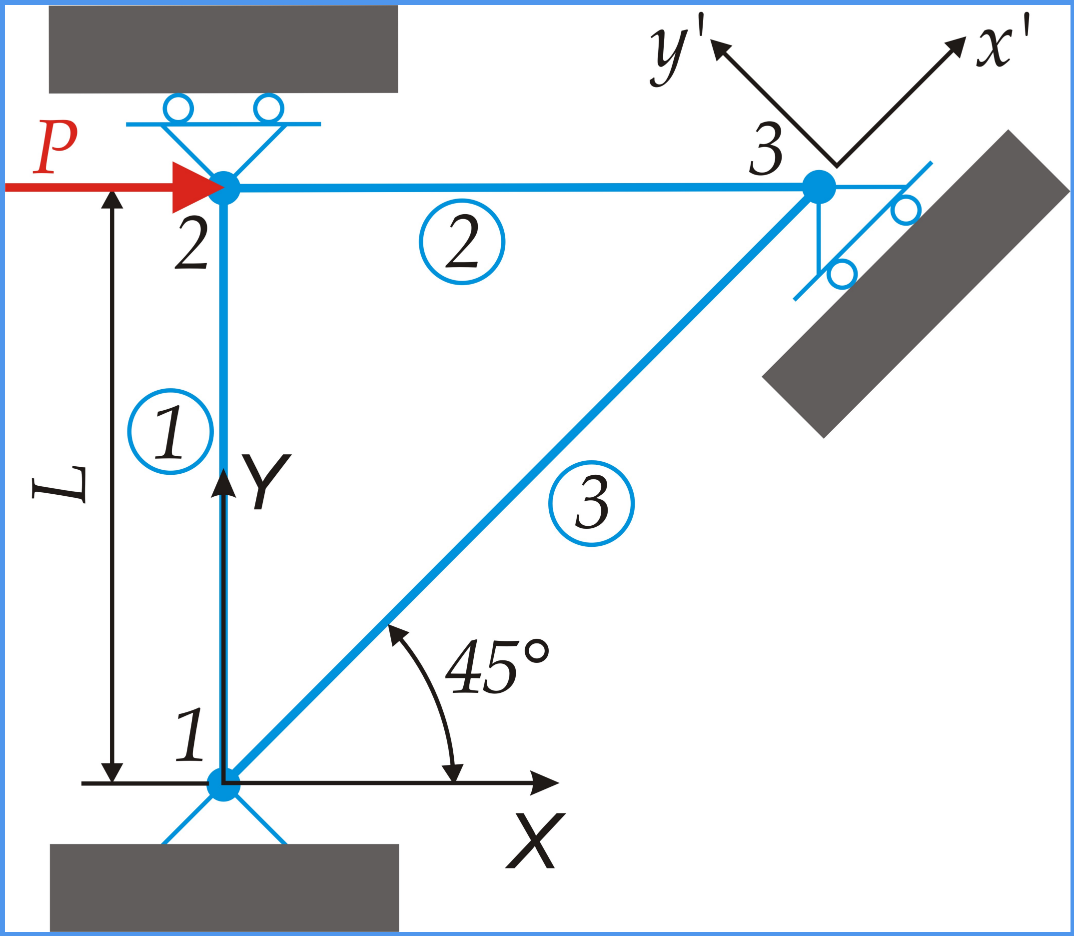


Problem: A simple plane truss is made of two identical bars (with E, A, and L), and loaded as shown in the figure. Find

1) displacement of node 2;

2) stress in each bar.

Example 7:



Problem: For the plane truss shown above,

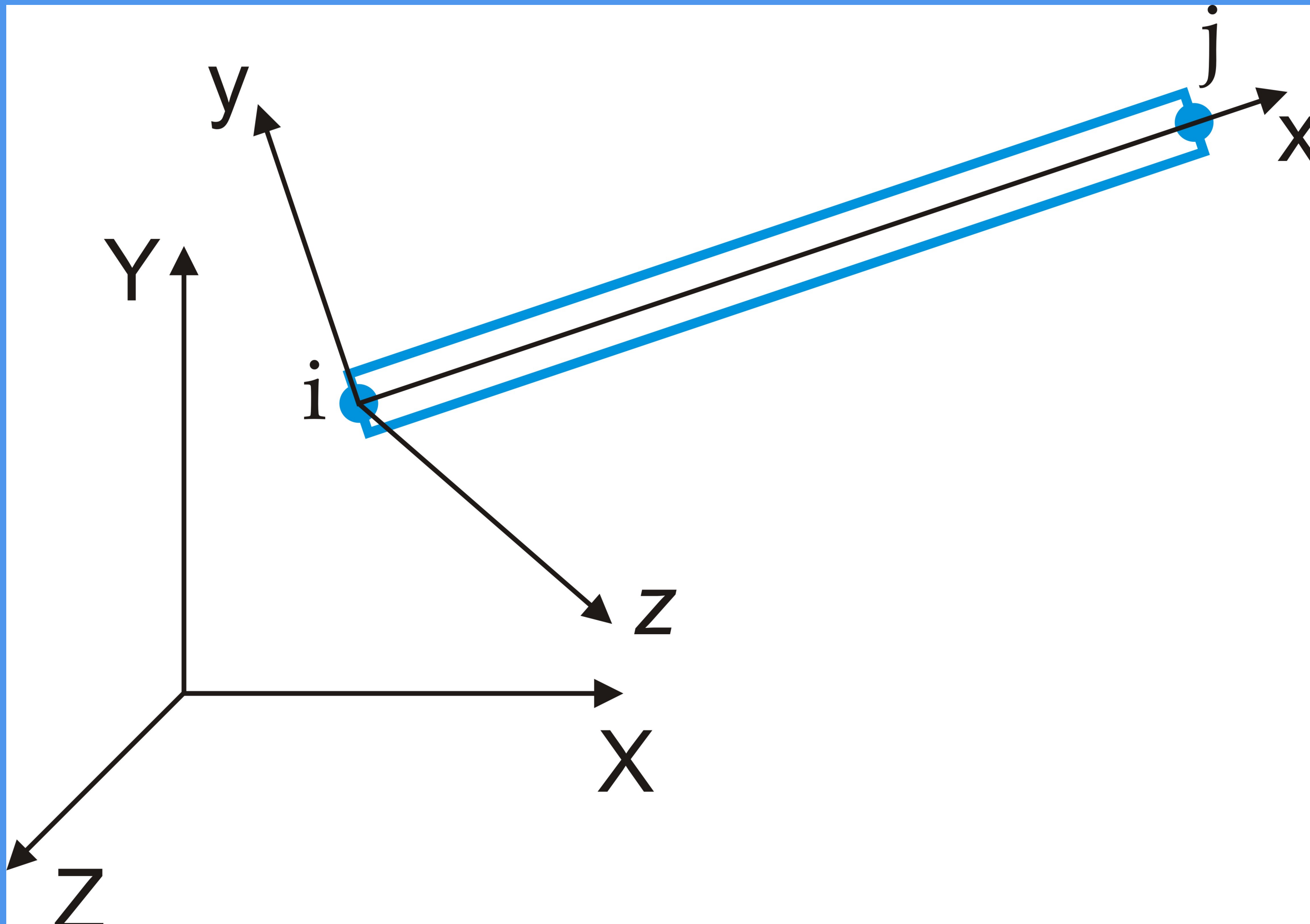
$$P = 1000\,\mathrm{kN} \qquad L = 1\,\mathrm{m},\ E = 210\,\mathrm{GPa},$$

$A = 6.0 \times 10^{-4}\mathrm{m}^2$ for elements 1 and 2,

$A = 2\sqrt{2} \times 10^{-4}\mathrm{m}^2$ for element 3.

Determine the displacements and reaction forces.

Bar Elements in 3-D Space



Local	Global
$x,\ y\ z$	$X,\ Y\ Z$
$u_i',\ v_i'\ w_i'$	$u_i,\ v_i\ w_i$
1 dof at node	3 dof's at node

Bar Elements in 3-D Space

Element stiffness matrice are calcualted in the local coordinate systems and then transformed into the global coordinate system $(X,\ Y,\ Z)$ where they are assembled.
FEA software will do this transformation automatically.

Input data for bar elements:

- $(X,\ Y,\ Z)$ for each node
- E and A for each element

back to start

Učebný text bol pripravený použitím
LaTeXu a balíka PPower4.