

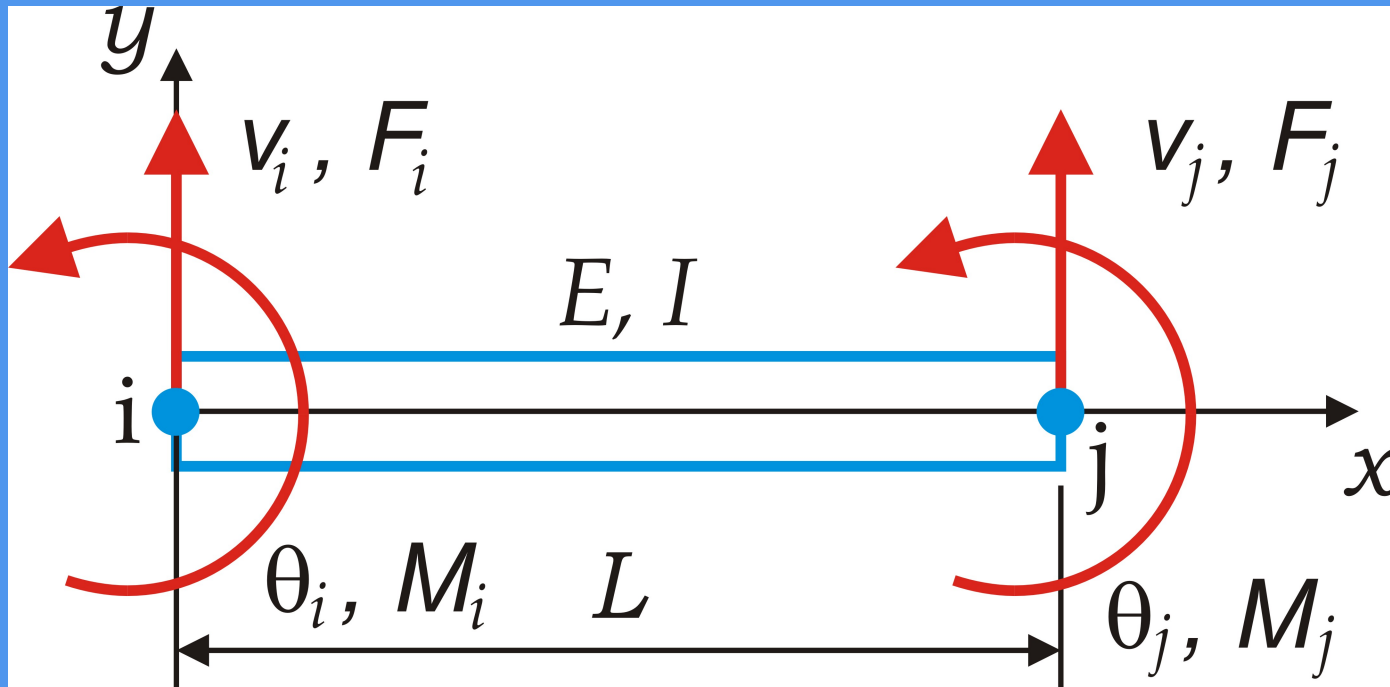
Učebný text (v anglickom jazyku)

NOSNÍKOVÝ PRVOK

prof. Ing. Roland Jančo, PhD.

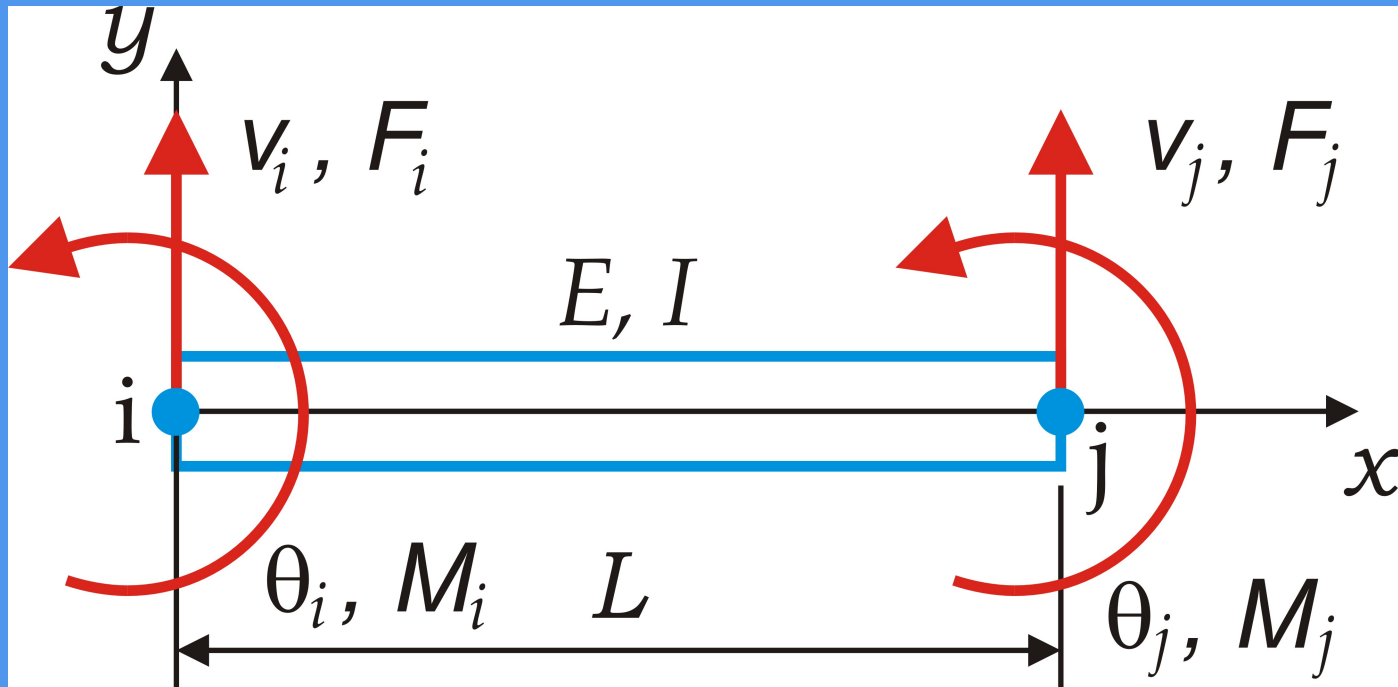
S podporou projektu KEGA 060STU-4/2016

Simple Plane Beam Element



L	length
I	moment of inertia of cross-section area
E	Young's modulus (elastic modulus)
$v = v(x)$	deflection (lateral displacement) of neutral axis
$\theta = \frac{dv}{dx}$	rotation about the z-axis
$F = F(x)$	shear force
$M = M(x)$	moment about the z-axis

Simple Plane Beam Element



Elementary Beam Theory

$$EI \frac{d^2 v}{dx^2} = M(x)$$

$$\sigma = -\frac{M y}{I}$$

Direct method

Nodal forces and moments are related to nodal displacements and rotations by the element stiffness matrix. We will formulate the element stiffness matrix using the direct approach, which **uses principles of superposition and equilibrium equations to find the terms in the stiffness matrix**. During our formulation of the element stiffness matrix for the one-dimensional beam element, *we assume that there are no axial displacements or forces*. There can be a vertical force and a moment applied at each node, which results in a vertical displacement and/or a rotation at each node. Since we are relating 4 nodal forces and moments, to 4 nodal displacements and rotations, the element stiffness matrix must be a 4×4 matrix, that relates each force and moment to the displacements and rotations.

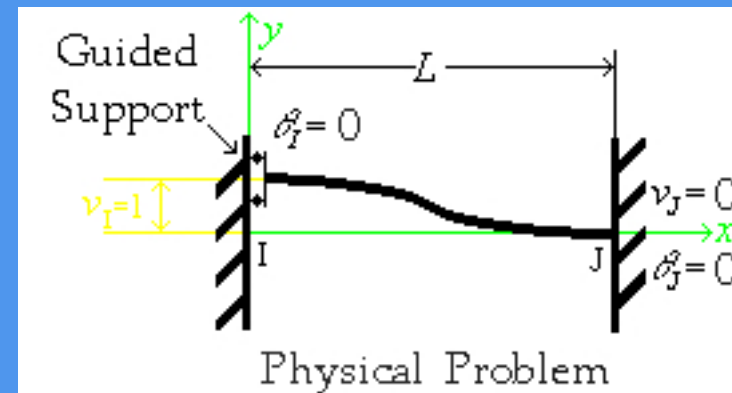
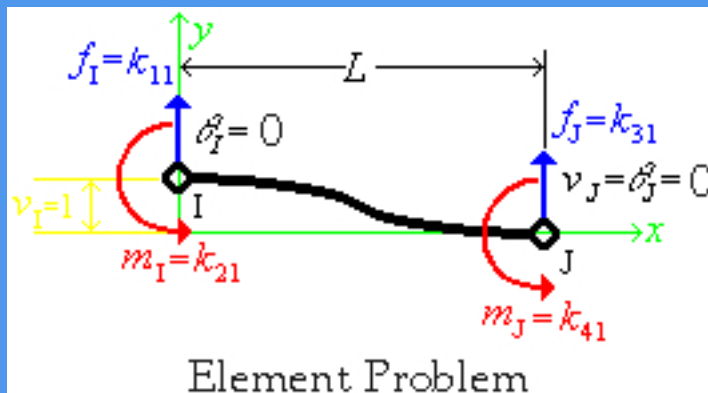
$$\begin{Bmatrix} f_I \\ m_I \\ f_J \\ m_J \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} v_I \\ \theta_I \\ v_J \\ \theta_J \end{Bmatrix}$$

In order to determine the coefficient values k_{ij} in the stiffness matrix, we will individually set each displacement and rotation equal to one, while keeping the others equal to zero. This is referred to as the unit displacement method. Each time we do this, we will be able to determine another column of the stiffness matrix. Each column of the element stiffness matrix is associated with an equilibrium problem.

Direct method

Column 1:

We start by setting the vertical displacement at node I equal to positive one, while keeping the vertical displacement at node J and the rotations at both nodes equal to zero. The element and physical problems associated with column 1 are shown below.



Our equation now becomes:

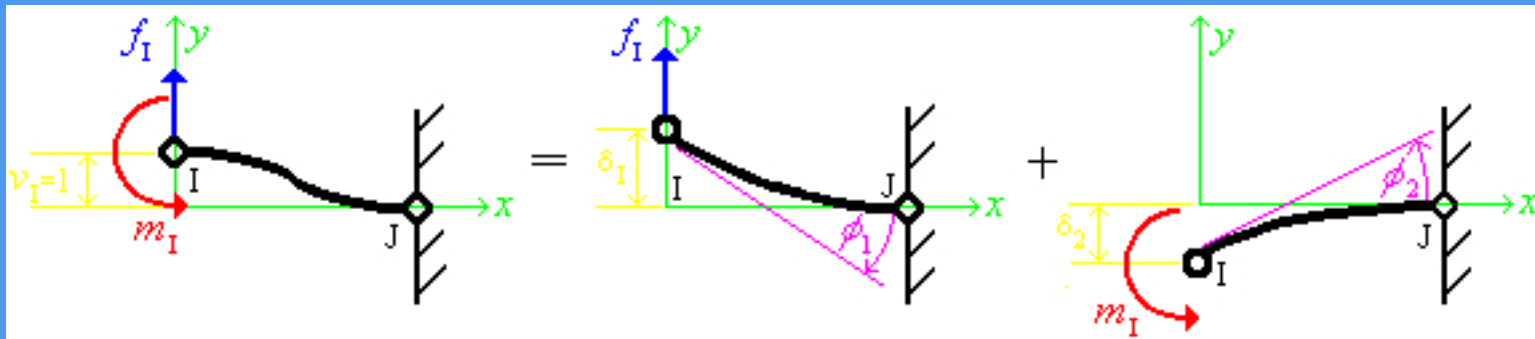
$$\begin{Bmatrix} f_I \\ m_I \\ f_J \\ m_J \end{Bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Direct method

and in expanded form we can derive the following relationships:

$$f_I = k_{11}, \quad m_I = k_{21}, \quad f_J = k_{31}, \quad m_J = k_{41},$$

Where the k values equal the forces and moments necessary, to cause the deformation state of the beam element. For a linear system, we can use the superposition principal to find these forces and moments. The principle of superposition sums the displacements and rotations of two or more cases in order to model a more complicated situation. In the figure below the sum of the rotations and in the two elements to the right of the "=" sign, are equal to the rotation of node I in the element problem at the left. Similarly the sum of the vertical displacements and of the two elements on the right are equal the total vertical displacement v_I at node I of the element on the left.

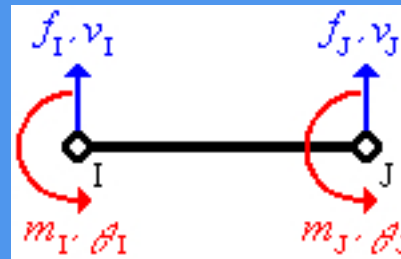


Direct method — Element problem using superposition

Using beam deflection and slope equations, (which can be found in most strength of materials texts) we can derive the following equations:

$$v_I = 1 = \delta_1 + \delta_2 \quad \text{or} \quad 1 = \frac{f_I L^3}{3EI} - \frac{m_I L^2}{2EI}$$

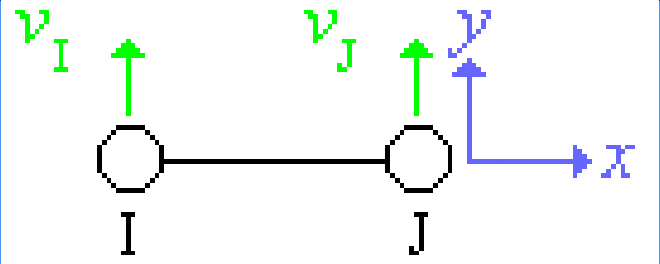
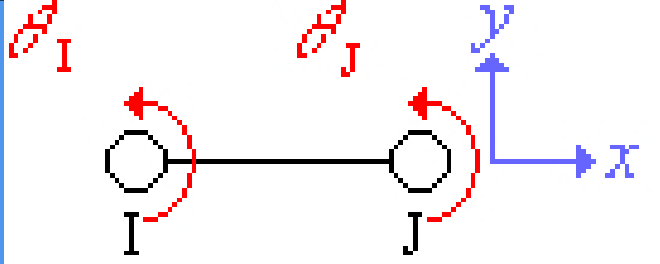
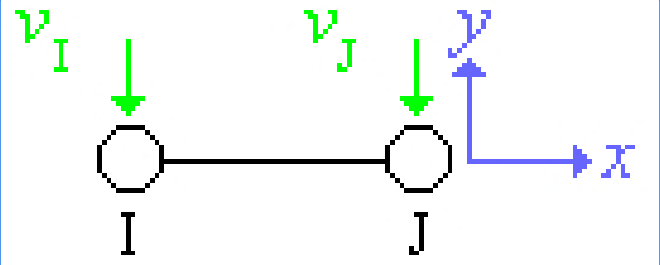
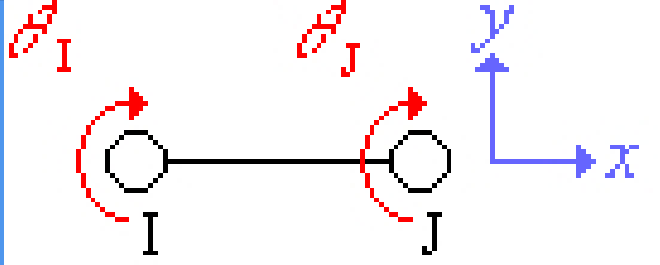
$$\theta_I = 0 = \phi_1 + \phi_2 \quad \text{or} \quad 0 = -\frac{f_I L^2}{2EI} + \frac{m_I L}{EI}$$



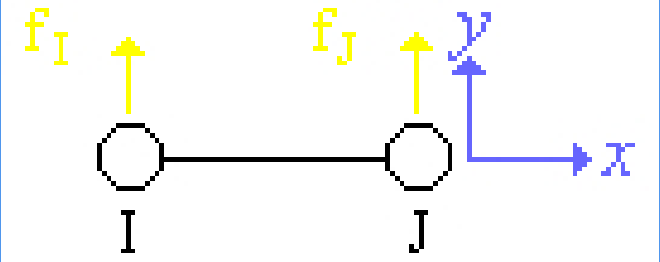
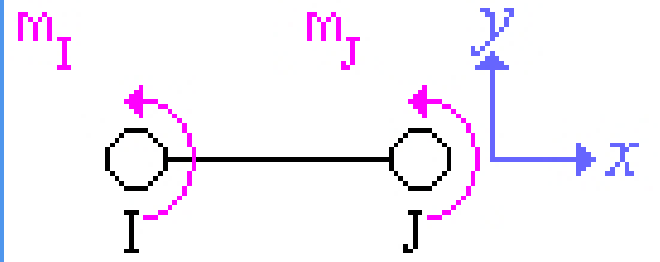
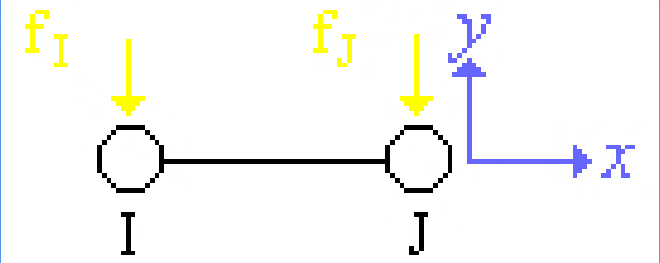
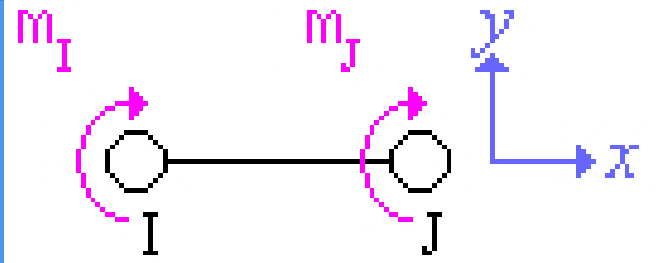
Where positive element *sign convention* has been used for nodal displacements and rotations as in the right-hand side above. Solving the two equations simultaneously for the force and moment at node I, results in the following equations:

$$f_I = \frac{12EI}{L^3} = k_{11}, \quad m_I = \frac{6EI}{L^2} = k_{21}.$$

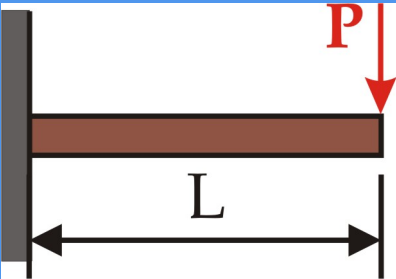
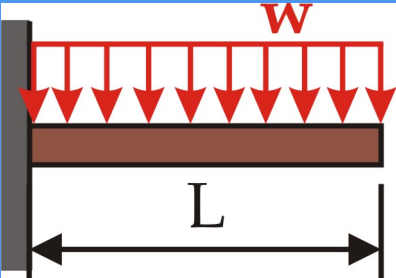
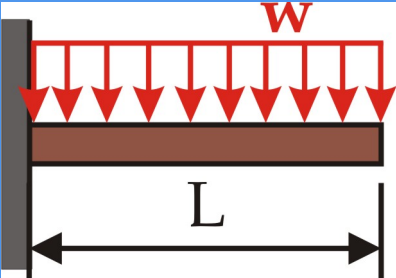
Direct method — Sign Convention

Solution Quantity	Traverse (Vertical) Displacement (v)	Rotation (Slope) (θ)
Common Units	Inches or Millimeters	Radians or Degrees
Nodal or Element Quantity	Nodal	Nodal
Positive Value(s)		
Negative Value(s)		

Direct method — Sign Convention

Solution Quantity	Shear Force (f)	Moment (m)
Common Units	Lbs or Newtons	Lb-In or Newton-Meter
Nodal or Element Quantity	Nodal	Nodal
Positive Value(s)		
Negative Value(s)		

Direct method — Deflection and Slope equations

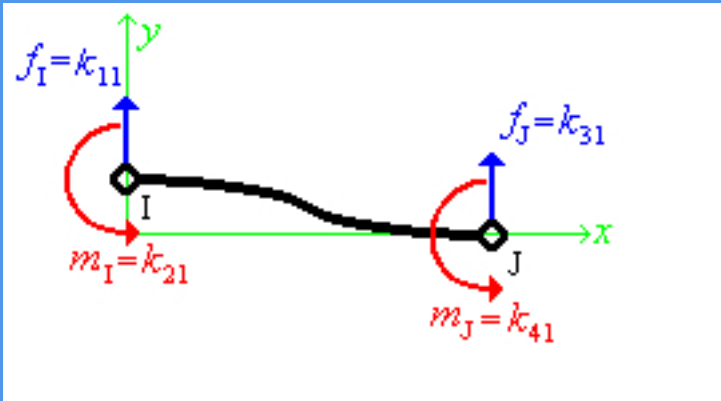
Beam and Loading	Maximum deflection	Slope at End	Equation of Elastic Curve
	$-\frac{PL^3}{3EI}$	$-\frac{PL^2}{2EI}$	$y = \frac{P}{6EI}(x^3 - 3Lx^2)$
	$-\frac{wL^4}{8EI}$	$-\frac{wL^3}{6EI}$	$y = -\frac{w}{24EI}(x^4 - 4Lx^3 + 6L^2x^2)$
	$-\frac{ML^2}{2EI}$	$-\frac{ML}{EI}$	$y = -\frac{M}{2EI}x^2$

Direct method — Deflection and Slope equations

Beam and Loading	Maximum deflection	Slope at End	Equation of Elastic Curve
<p>A simply supported beam of length L with a point load P applied at the center, $L/2$ from the left support.</p>	$-\frac{PL^3}{48EI}$	$\pm \frac{PL^2}{16EI}$	<p>For $x \leq L/2$:</p> $y = \frac{P}{48EI}(4x^3 - 3L^2x)$
<p>A simply supported beam of length L with a uniformly distributed load w applied downwards along the entire length.</p>	$-\frac{5wL^4}{384EI}$	$\pm \frac{wL^3}{24EI}$	$y = -\frac{w}{24EI}(x^4 - 2Lx^3 + L^3x)$
<p>A simply supported beam of length L with a clockwise moment M applied at the right end, labeled B. The left end is labeled A.</p>	$-\frac{ML^2}{9\sqrt{3}EI}$	$\theta_A = +\frac{ML}{6EI}$ $\theta_B = -\frac{ML}{3EI}$	$y = -\frac{M}{6EI}(x^3 - L^2x)$

Direct method

Next we use equilibrium equations and the usual sign conventions, setting the sum of the vertical forces and the sum of the moments each equal to zero, in order to find the last two terms in column one:



$$+ \uparrow \sum F_y = 0$$

$$f_J = -f_I = \frac{-12EI}{L^3} = k_{31}$$

$$+ \sum M_Z = 0 \quad \text{about node J}$$

$$m_J = -(m_I - f_I L) = \frac{6EI}{L^2} = k_{41}$$

Direct method

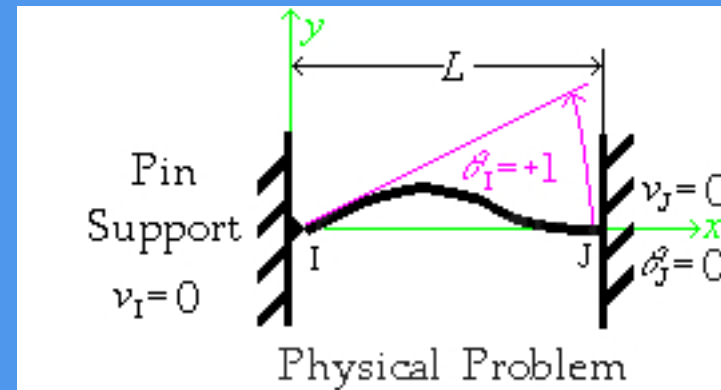
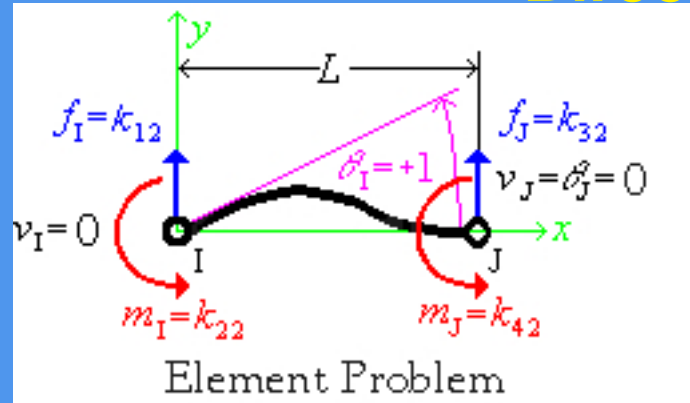
The terms for the first column of the element stiffness matrix are shown below :

$$\begin{Bmatrix} f_I \\ m_I \\ f_J \\ m_J \end{Bmatrix} = \begin{bmatrix} \frac{12EI}{L^3} & k_{12} & k_{13} & k_{14} \\ \frac{6EI}{L^2} & k_{22} & k_{23} & k_{24} \\ -\frac{12EI}{L^3} & k_{32} & k_{33} & k_{34} \\ \frac{6EI}{L^2} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} v_I \\ \theta_I \\ v_J \\ \theta_J \end{Bmatrix}$$

Column 2:

In order to obtain the second column we assume a rotation at node **I** that is equal to positive one, while keeping both vertical displacements and the rotation at node **J** equal to zero.

Direct method



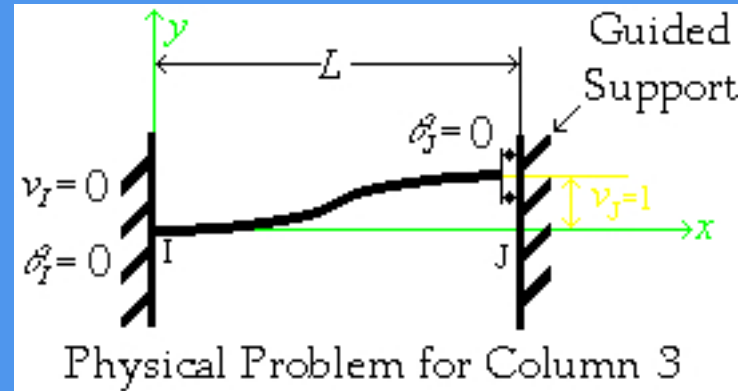
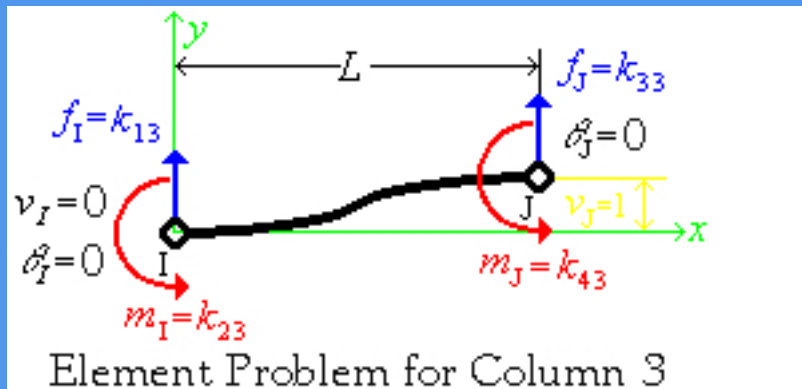
Using the same procedure as column 1, we find that the terms of the second column are:

$$\begin{Bmatrix} f_I \\ m_I \\ f_J \\ m_J \end{Bmatrix} = \begin{bmatrix} k_{11} & \frac{6EI}{L^2} & k_{13} & k_{14} \\ k_{21} & \frac{4EI}{L} & k_{23} & k_{24} \\ k_{31} & -\frac{6EI}{L^2} & k_{33} & k_{34} \\ k_{41} & \frac{2EI}{L} & k_{43} & k_{44} \end{bmatrix} \begin{Bmatrix} v_I \\ \theta_I \\ v_J \\ \theta_J \end{Bmatrix}$$

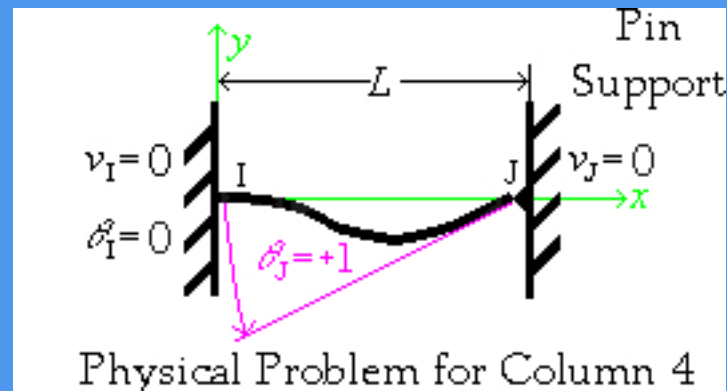
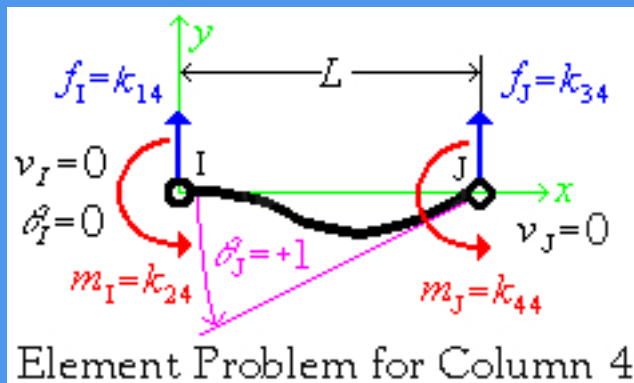
Direct method

Columns 3 and 4: We can find the terms in the third and fourth columns, by individually setting the vertical displacement and rotation at node J equal to one, as we did for node I. The deformation states for these problems are shown below:

Deformation state for column 3



Deformation state for column 4



Direct method

When the common terms are factored out, we end up with the following stiffness matrix:

$$\mathbf{k} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

Our FEM equation for the one-dimensional beam element becomes:

$$\begin{Bmatrix} f_I \\ m_I \\ f_J \\ m_J \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} v_I \\ \theta_I \\ v_J \\ \theta_J \end{Bmatrix}$$

Formal Approach

Apply the formula,

$$\mathbf{k} = \int_0^L \mathbf{B}^T EI \mathbf{B} dx$$

To derive this, we introduce the shape functions,

$$\begin{aligned} N_1(x) &= 1 - 3x^2/L^2 + 2x^3/L^3 \\ N_2(x) &= x - 2x^2/L + x^3/L^2 \\ N_3(x) &= 3x^2/L^2 - 2x^3/L^3 \\ N_4(x) &= -x^2/L + x^3/L^2 \end{aligned}$$

Then, we can represent the deflection as,

$$\begin{aligned} v(x) &= \mathbf{N} \mathbf{u} = \\ &= \begin{bmatrix} N_1(x) & N_2(x) & N_3(x) & N_4(x) \end{bmatrix} \begin{Bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{Bmatrix} \end{aligned}$$

Formal Approach

which is a cubic function. Notice that,

$$\begin{aligned} N_1 + N_3 &= 1 \\ N_2 + N_3L + N_4 &= x \end{aligned}$$

which implies that the rigid body motion is represented by the assumed deformed shape of the beam.

Curvature of the beam is,

$$\frac{d^2v}{dx^2} = \frac{d^2}{dx^2} \mathbf{N} \mathbf{u} = \mathbf{B} \mathbf{u}$$

where the strain-displacement matrix \mathbf{B} is given by,

$$\begin{aligned} \mathbf{B} &= \frac{d^2}{dx^2} \mathbf{N} = \left[N_1''(x) \quad N_2''(x) \quad N_3''(x) \quad N_4''(x) \right] \quad (2) \\ &= \left[-\frac{6}{L^2} + \frac{12x}{L^3} \quad -\frac{4}{L} + \frac{6x}{L^2} \quad \frac{6}{L^2} - \frac{12x}{L^3} \quad -\frac{2}{L} + \frac{6x}{L^2} \right] \end{aligned}$$

Formal Approach

Strain energy stored in the beam element is

$$\begin{aligned}
 U &= \frac{1}{2} \int_V \boldsymbol{\sigma}^T \boldsymbol{\varepsilon} dV = \frac{1}{2} \int_0^L \int_A \left(-\frac{M y}{I} \right) \frac{1}{E} \left(-\frac{M y}{I} \right) dA dx = \\
 &= \frac{1}{2} \int_0^L M^T \frac{1}{EI} M dx = \frac{1}{2} \int_0^L \left(\frac{d^2 v}{dx^2} \right)^T EI \left(\frac{d^2 v}{dx^2} \right) dx = \\
 &= \frac{1}{2} \int_0^L (\mathbf{B} \mathbf{u})^T EI (\mathbf{B} \mathbf{u}) dx = \\
 &= \frac{1}{2} \mathbf{u}^T \left(\int_0^L \mathbf{B}^T EI \mathbf{B} dx \right) \mathbf{u}.
 \end{aligned}$$

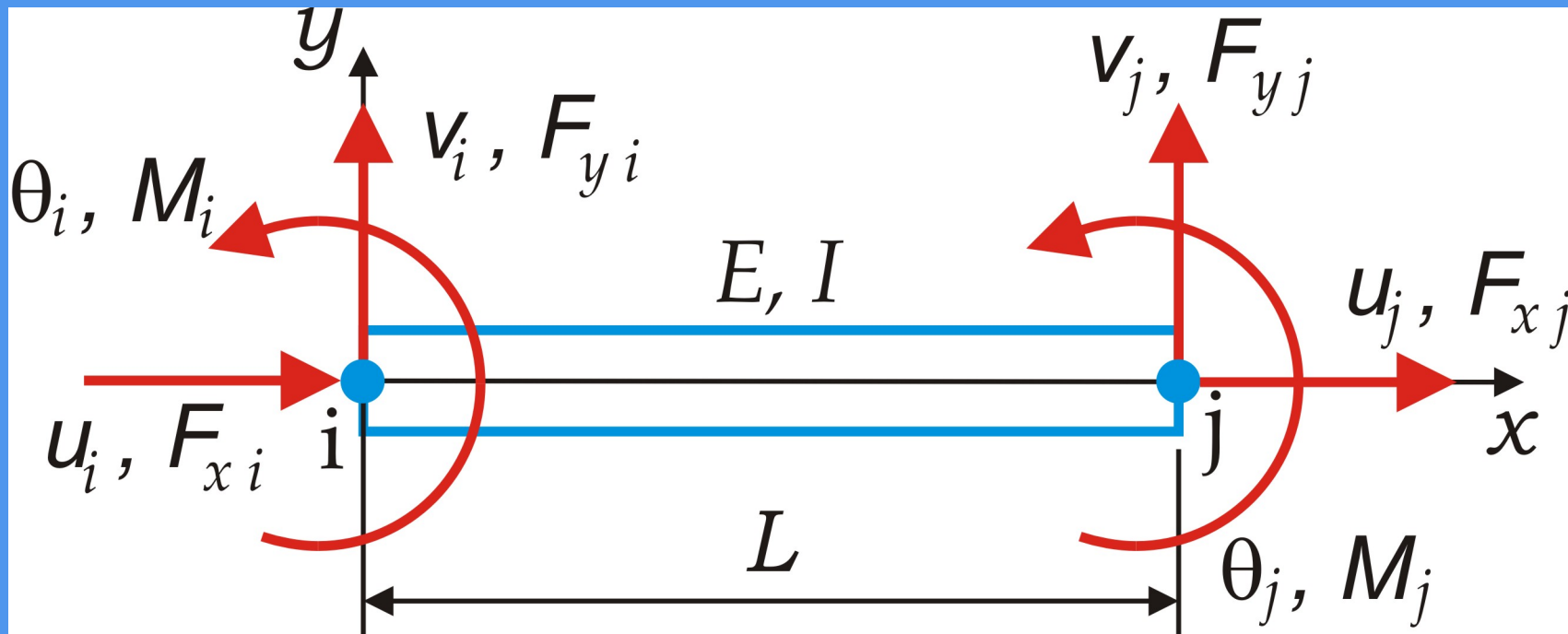
We conclude that the stiffness matrix for the simple beam element is

$$\mathbf{k} = \int_0^L \mathbf{B}^T EI \mathbf{B} dx.$$

Formal Approach

Applying the result in (2) and carrying out the integration, we arrive at the same stiffness matrix as given in direct approach,

$$\frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} v_i \\ \theta_i \\ v_j \\ \theta_j \end{Bmatrix} = \begin{Bmatrix} F_i \\ M_i \\ F_j \\ M_j \end{Bmatrix}$$



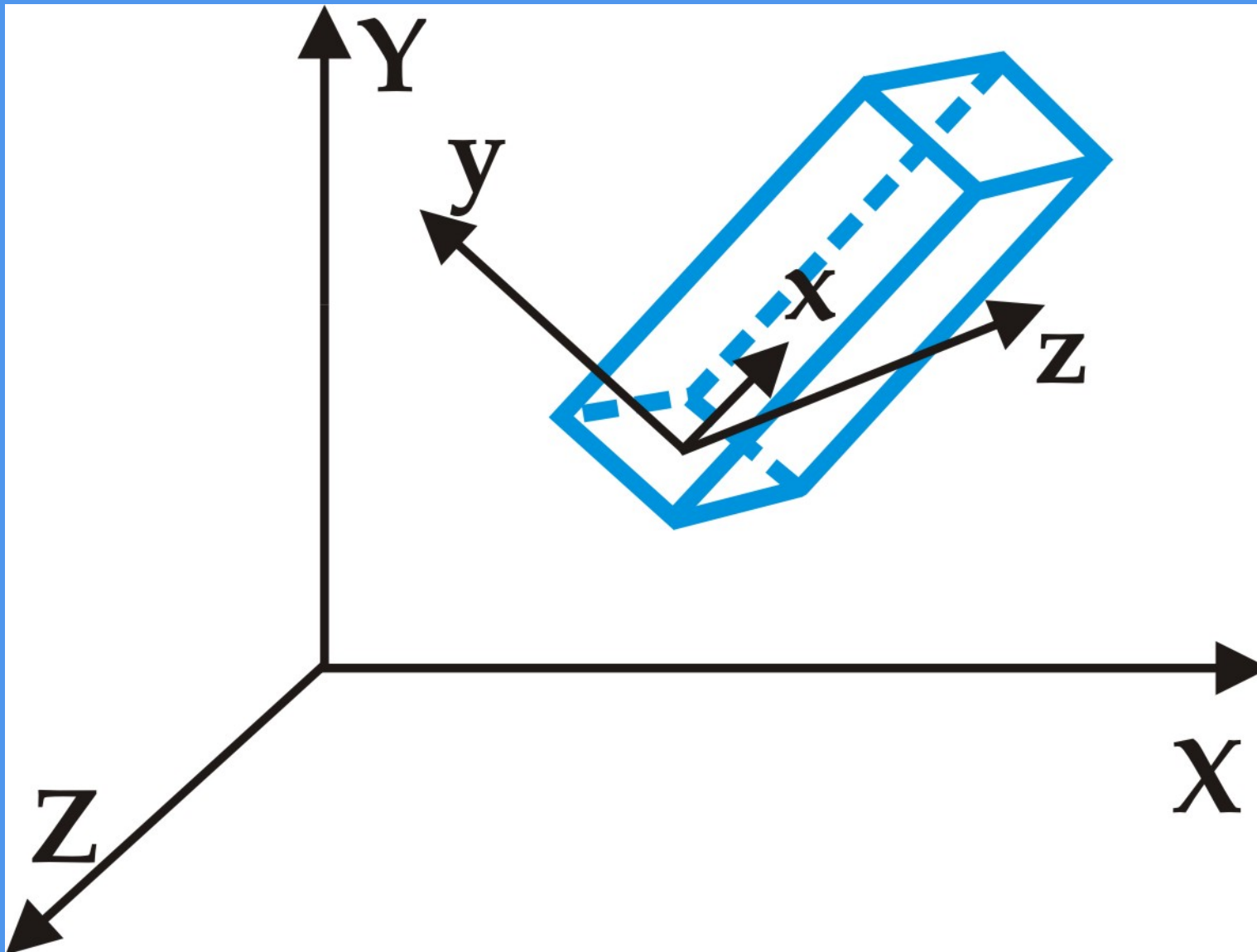
Formal Approach

Combining the axial stiffness (bar element), we obtain the stiffness matrix of a **general 2-D beam element**,

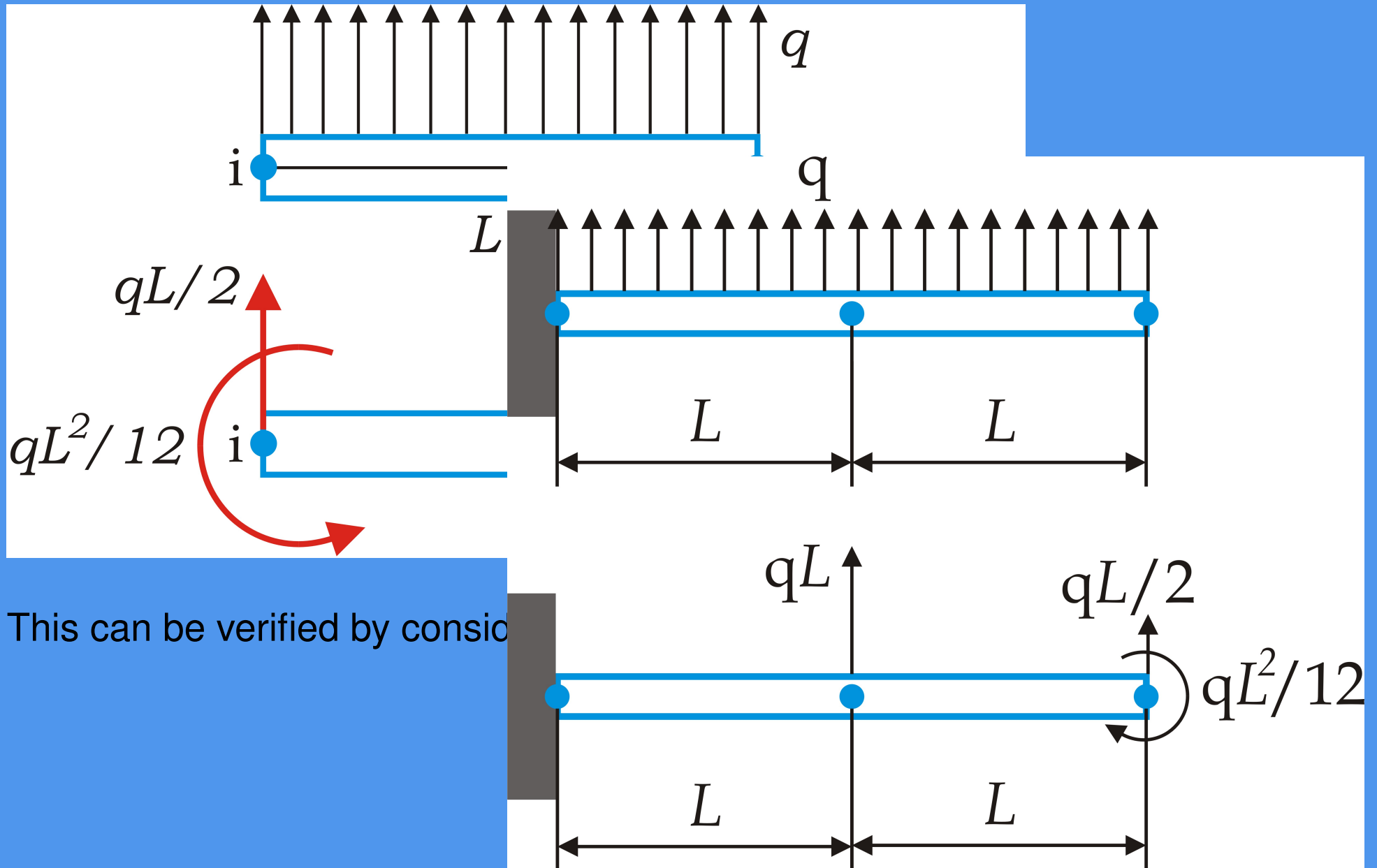
$$\begin{bmatrix}
 \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\
 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\
 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\
 -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\
 0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\
 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L}
 \end{bmatrix}
 \begin{Bmatrix}
 u_i \\
 v_i \\
 \theta_i \\
 u_j \\
 v_j \\
 \theta_j
 \end{Bmatrix}
 =
 \begin{Bmatrix}
 F_{xi} \\
 F_{yi} \\
 M_i \\
 F_{xj} \\
 F_{yj} \\
 M_j
 \end{Bmatrix}$$

3-D Beam Element

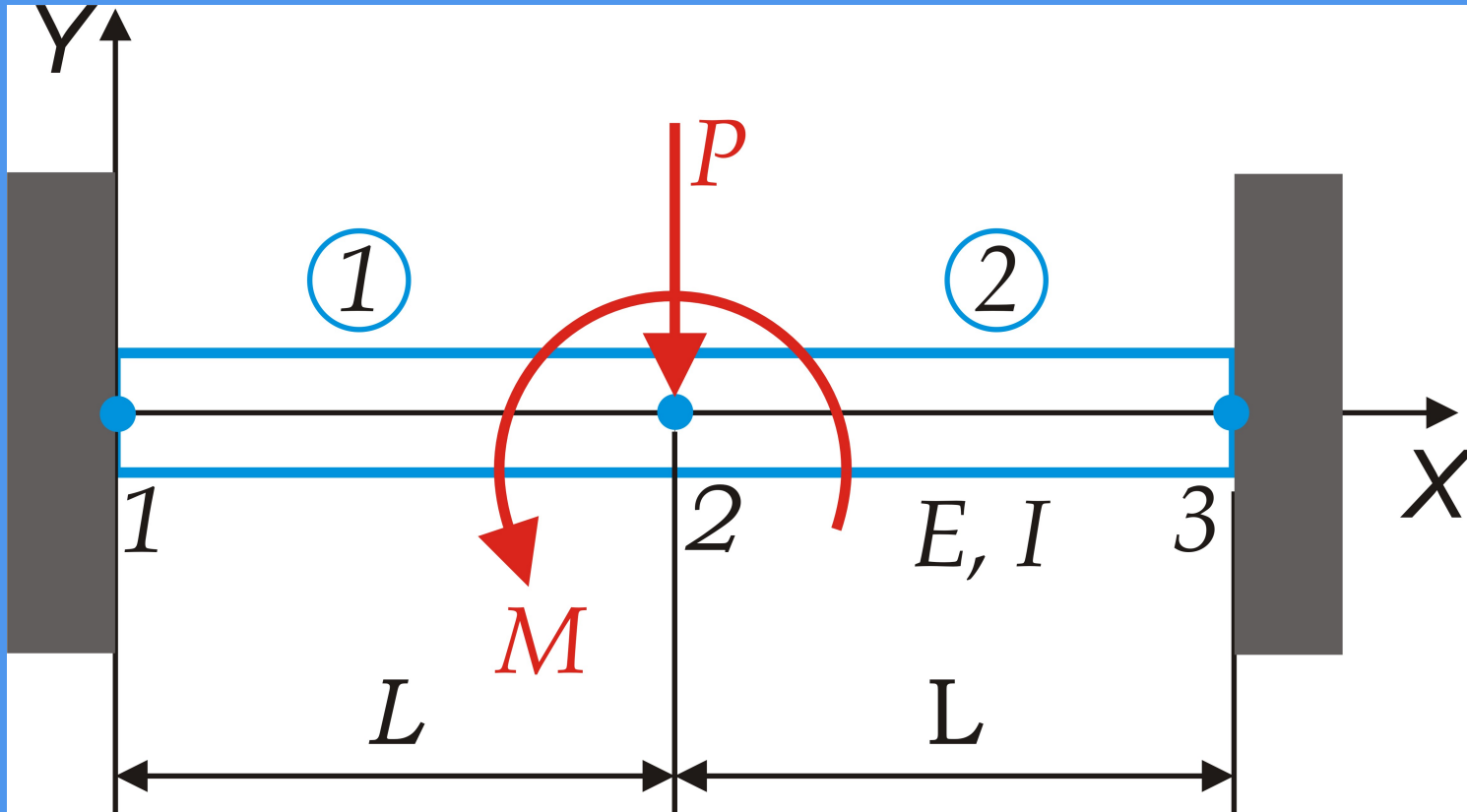
The element stiffness matrix is formed in the local (2-D) coordinates system first and then transformed into the global (3-D) coordinate system to be assembled.



Equivalent Nodal Loads of Distributed Transverse Load



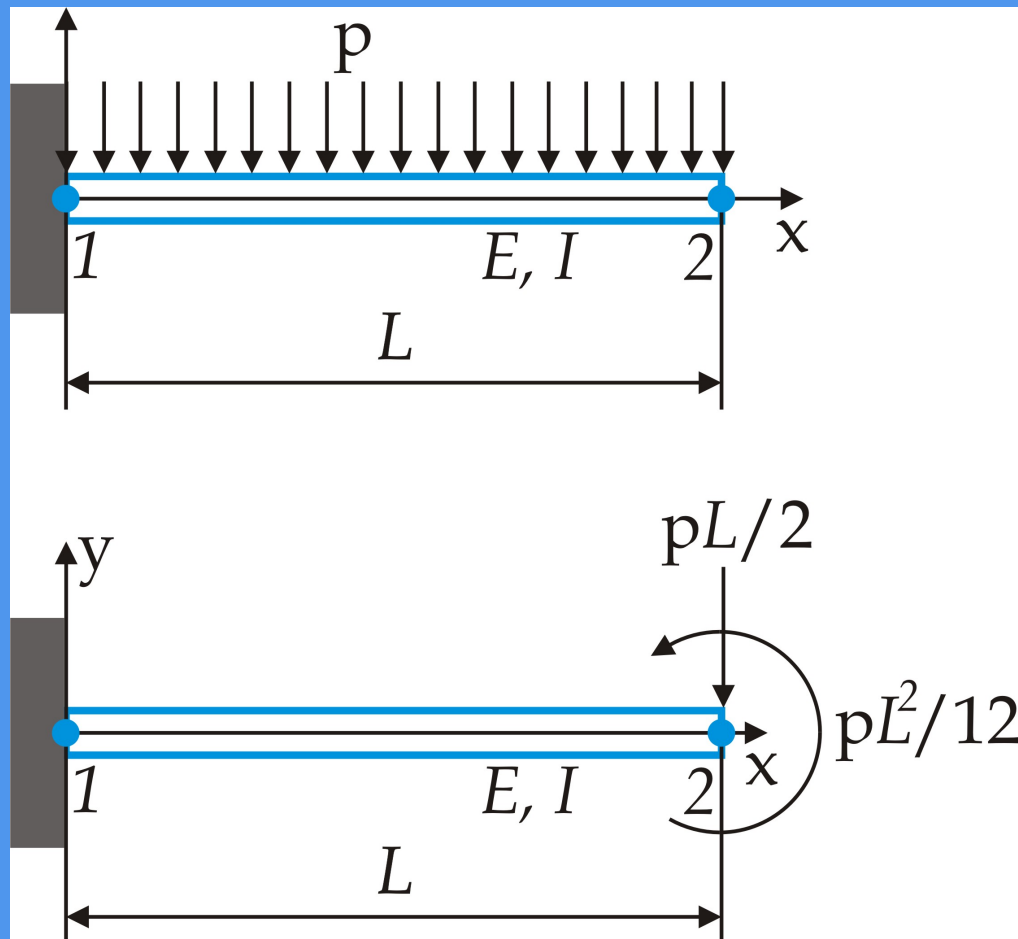
Example 1



Given: The beam shown above is clamped at the two ends and acted upon by the force P and moment M in the middle.

Find: The deflection and rotation at the center node and the reaction forces and moments at the two ends.

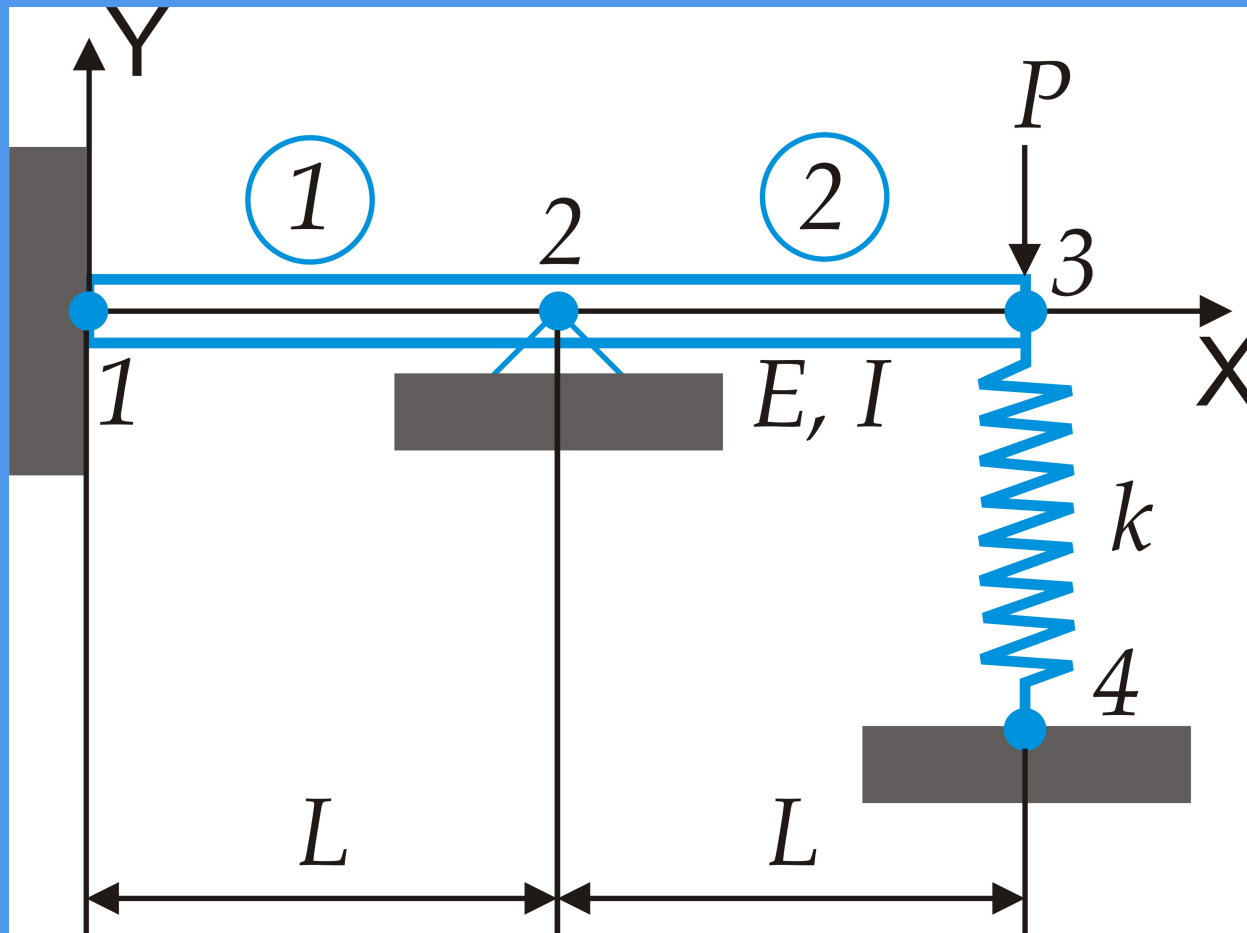
Example 2



Given: A cantilever beam with distributed lateral load p as shown above.

Find: The deflection and rotation at the right end, the reaction forces and moments at the left end.

Example 3



Given:

$$P = 50 \text{ kN}, \quad k = 200 \text{ kN/m}, \quad L = 3 \text{ m},$$

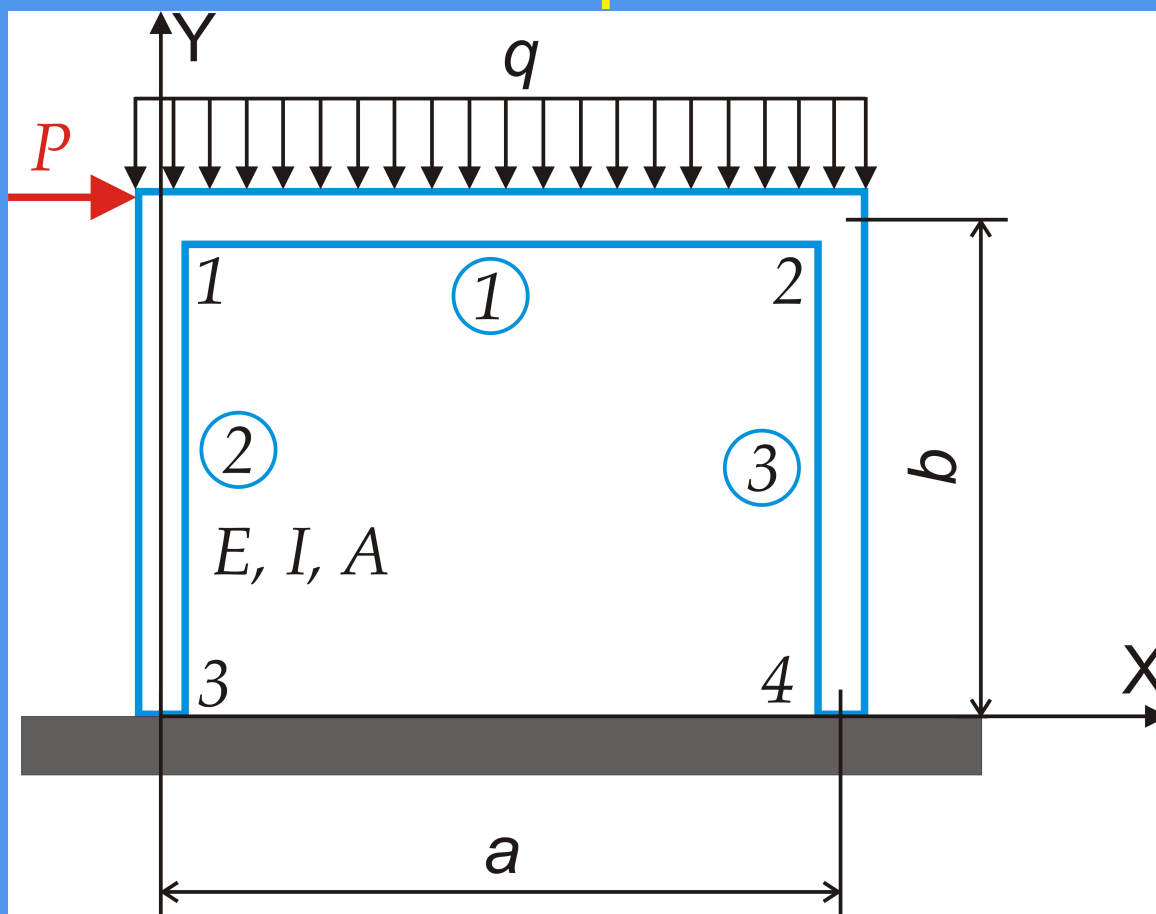
$$E = 210 \text{ GPa}, \quad I = 2 \times 10^{-4} \text{ m}^4.$$

Find: Deflections, rotations and reaction forces.

FE Analysis of Frame Structures

Members in a frame are considered to be rigidly connected. Both forces and moments can be transmitted through their joints. We need the **general beam element** (combinations of bar and simple beam elements) to model frames.

Example 4



Given:

$$E = \quad \text{MPa} \quad I = \quad \text{mm}^4 \quad A = \quad \text{mm}^2$$

Find: Displacements and rotations of two joints 1 and 2.

back to start



Učebný text bol pripravený použitím
 $\text{L}^{\text{A}}\text{T}_{\text{E}}\text{X}$ u a balíka PPower4.

