Učebný text (v anglickom jazyku)

R I E Š E N I E 2D A 3D Ú L O H M K P

prof. Ing. Roland Jančo, PhD.

S podporou projektu KEGA 060STU-4/2016

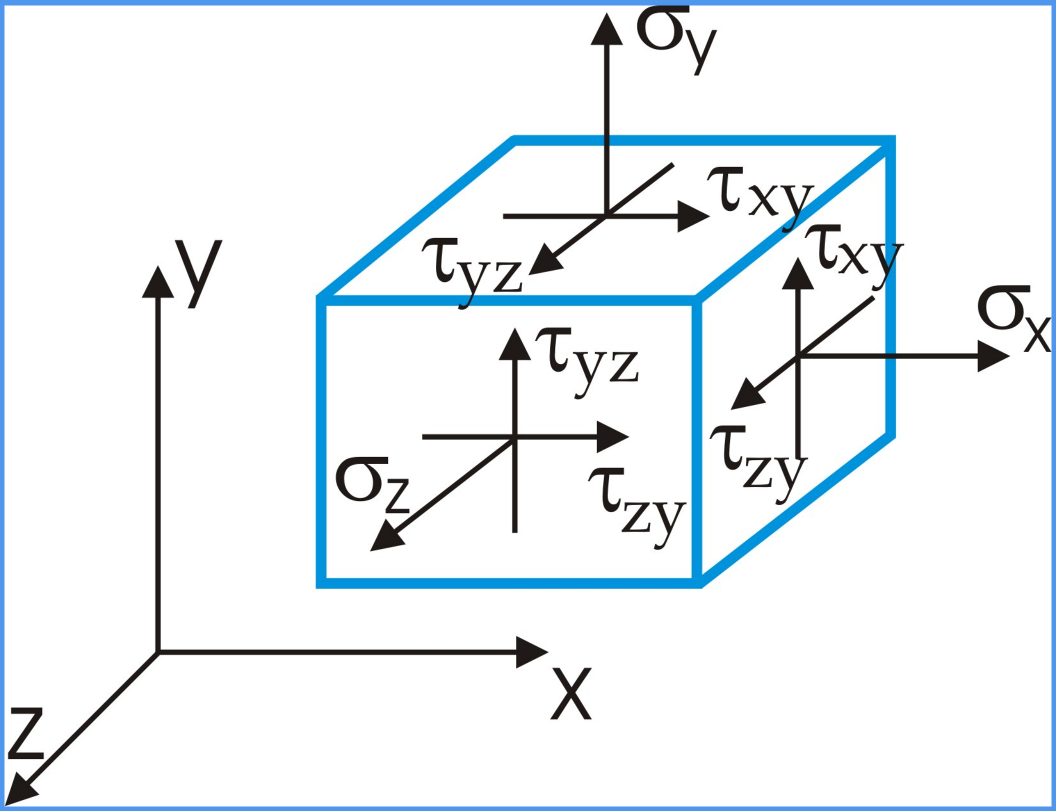
Review of the Basic Theory

In general, the stresses and strains in a structure consist of six components

$$\sigma_x, \quad \sigma_y, \quad \sigma_z, \quad \tau_{xy}, \quad \tau_{yz} \quad \tau_{zx} \qquad \text{for stresses}$$

and

$$\varepsilon_x, \quad \varepsilon_y, \quad \varepsilon_z, \quad \gamma_{xy}, \quad \gamma_{yz}, \quad \gamma_{zx} \qquad \text{for strains.}$$



Review of the Basic Theory

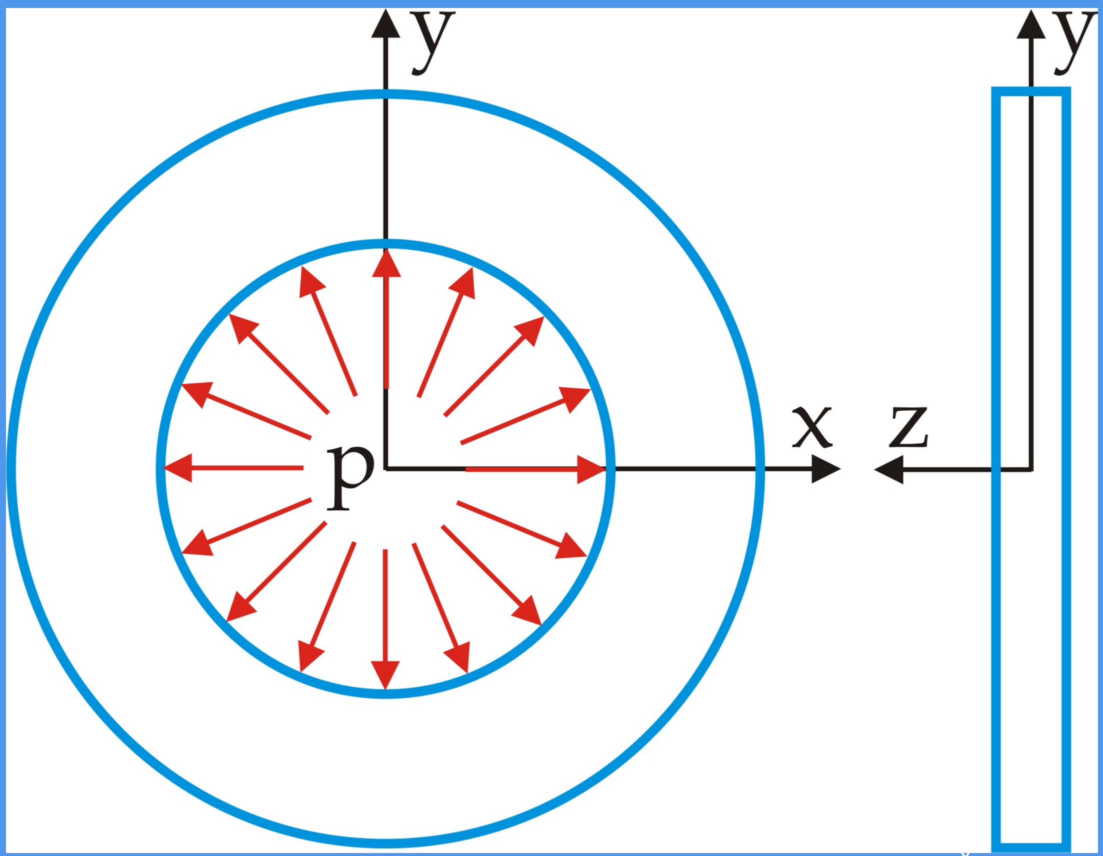
Under contain conditions, the state of stresses and strains can be simplified. A general 3-D structure analysis can, therefore, be reduced to a 2-D analysis.

Plane (2-D) Problems

1) Plane stress:

$$\sigma_z = \tau_{yz} = \tau_{zx} = 0 \qquad\qquad (\varepsilon_z \neq 0)$$

A thin plane structure with constant thickness and loading within the plane of the structure (xy-plane).

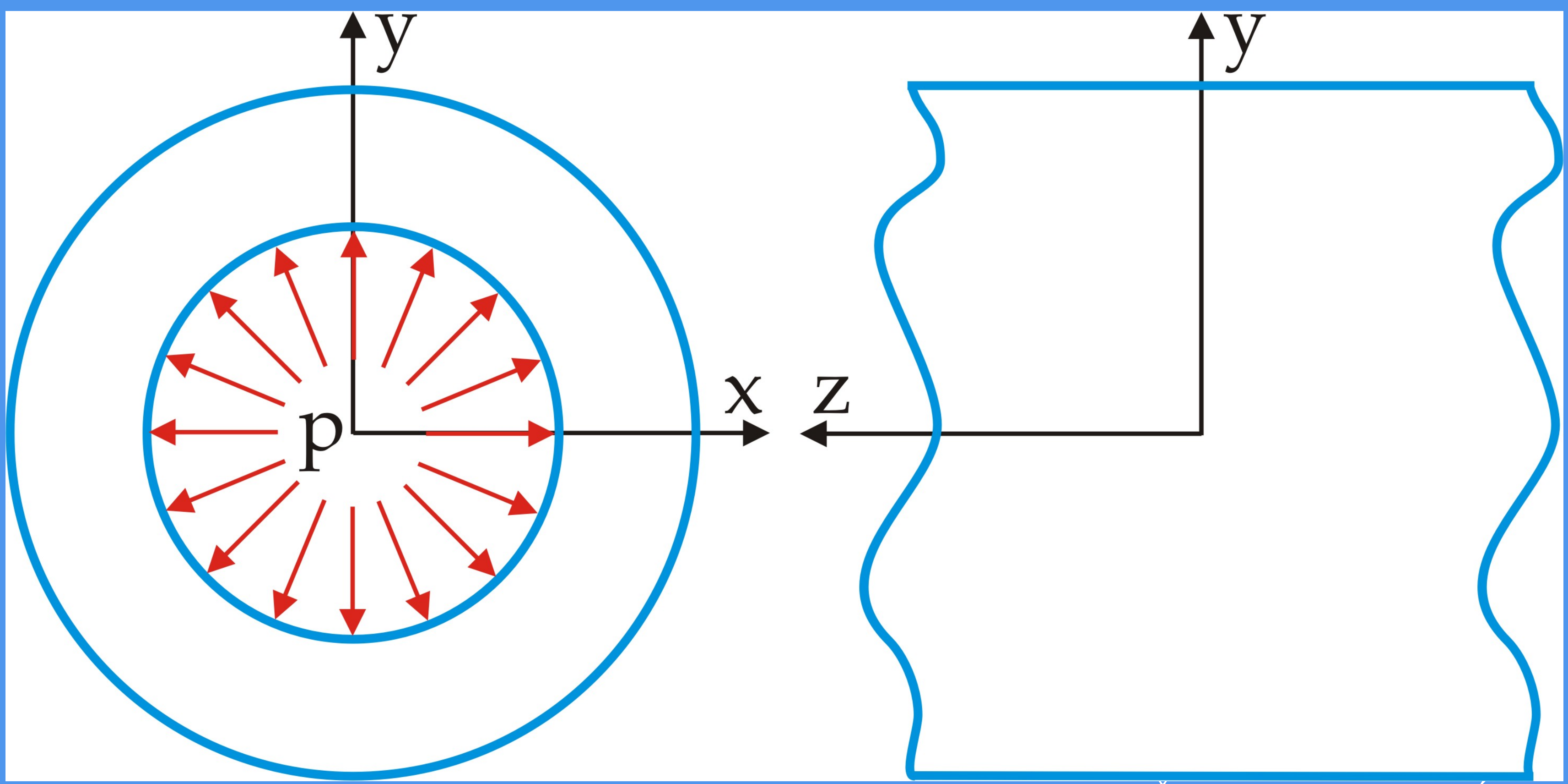


Plane (2-D) Problems

2) Plane strain:

$$\varepsilon_z = \gamma_{yz} = \gamma_{zx} = 0 \qquad\qquad (\sigma_z \neq 0)$$

A long structure with a uniform cross section and transverse loading along its length (z-direction).



Stress-Strain-Temperature (Constitutive) Relations

For elastic and isotropic materials, we have,

$$\left\{ \begin{array}{c} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{array} \right\} = \left[\begin{array}{ccc} 1/E & -\nu/E & 0 \\ -\nu/E & 1/E & 0 \\ 0 & 0 & 1/G \end{array} \right] \left\{ \begin{array}{c} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{array} \right\} = \left\{ \begin{array}{c} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \varepsilon_{xy0} \end{array} \right\}$$

or

$$\varepsilon = \mathbf{E}^{-1}\sigma + \varepsilon_0$$

where ε_0 is the initial strain, E the Young's modulus, ν the Poisson's ratio and G the shear modulus. Note that,

$$G = \frac{E}{2(1+\nu)}$$

which means that there are only two independent materials constants for *homogeneous* and *isotropic* materials.

Stress-Strain-Temperature (Constitutive) Relations

We can also express stress in terms of strains by solving the above equation,

$$\left\{ \begin{array}{c} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{array} \right\} = \frac{E}{1-\nu^2} \left[\begin{array}{ccc} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{array} \right] \left(\left\{ \begin{array}{c} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{array} \right\} - \left\{ \begin{array}{c} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \gamma_{xy0} \end{array} \right\} \right)$$

or,

$$\sigma = \mathbf{E}\varepsilon + \sigma_0$$

where $\sigma_0 = -\mathbf{E}\varepsilon_0$ is the initial stress.
The above relations are valid for *plane stress* case. For *plain strain* case, we need to replace the material constants in the above equations in the following fashion,

$$\begin{aligned} E &\rightarrow \frac{E}{1-\nu^2} \\ \nu &\rightarrow \frac{\nu}{1-\nu} \\ G &\rightarrow G \end{aligned}$$

Stress-Strain-Temperature (Constitutive) Relations

For example, the stress is related to strain by

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix} \left(\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} - \begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \gamma_{xy0} \end{Bmatrix} \right)$$

in the *plane strain* case.
Initial strains due to *temperature change* (thermal loading) is given by,

$$\begin{Bmatrix} \varepsilon_{x0} \\ \varepsilon_{y0} \\ \gamma_{xy0} \end{Bmatrix} = \begin{Bmatrix} \alpha\Delta T \\ \alpha\Delta T \\ 0 \end{Bmatrix}$$

where α is the coefficient of thermal expansion, ΔT the change of temperature.
Note that if the structure is free to deform under thermal loading, there will be no (elastic) stresses in the structure.

Strain and Displacement Relations

For small strains and small rotations, we have,

$$\varepsilon_x = \frac{\partial u}{\partial x}, \qquad \varepsilon_y = \frac{\partial v}{\partial y}, \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

In matrix form,

$$\left\{ \begin{array}{c} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{array} \right\} = \left[\begin{array}{cc} \partial / \partial x & 0 \\ 0 & \partial / \partial y \\ \partial / \partial y & \partial / \partial x \end{array} \right] \left\{ \begin{array}{c} u \\ v \end{array} \right\}, \qquad \text{or} \qquad \boldsymbol{\varepsilon} = \mathbf{D}\mathbf{u}$$

From this relation, we know that the strains (and thus stresses) are one order lower than the displacements, if the displacements are represented by polynomials.

Equilibrium Equations

In elasticity theory, the stresses in the structure must satisfy the following equilibrium equations,

$$\begin{aligned} \frac{\partial\,\sigma_x}{\partial\,x} + \frac{\partial\,\tau_{xy}}{\partial\,y} + f_x &= 0 \\ \frac{\partial\,\tau_{xy}}{\partial\,x} + \frac{\partial\,\sigma_y}{\partial\,y} + f_y &= 0 \end{aligned} \tag{1}$$

where f_x and f_y are body forces (such as gravity forces) per unit volume. In FEM, these equilibrium conditions are satisfied in an approximate sense.

Boundary condition

The boundary S of the body can be divided into two parts, S_u and S_t. The boundary conditions (BC's) are described as,

$$\begin{aligned} u &= \bar{u}, & v &= \bar{v}, & &\text{on } S_u \\ t_x &= \bar{t}_x, & t_y &= \bar{t}_y, & &\text{on } S_t \end{aligned} \tag{2}$$

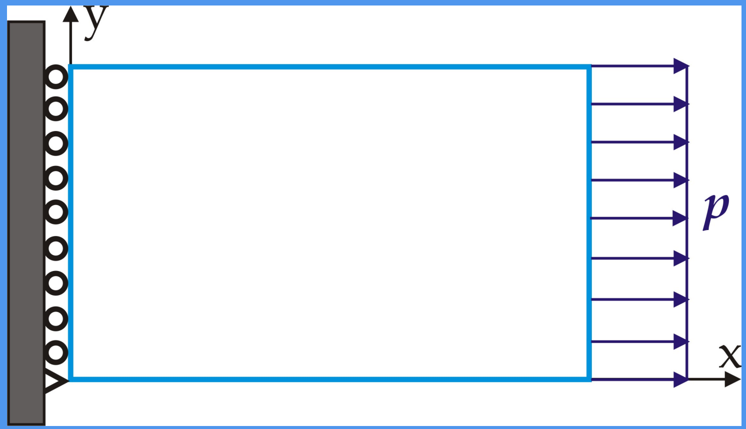
in which t_x and t_y are traction forces (stresses on the boundary) and the barred quantities are those with known values.
In FEM, all types of loads (distributed surface loads, body forces, concentrated forces and moments, etc.) are converted to points forces acting at the nodes.

Exact Elasticity Solution

The exact solution (displacements, strains and stress) of a given problem must satisfy the equlibrium equations (2), the given boundary conditions (3) and compatibility conditions (structures should deform in a continuous manner, no cracks and overlaps in the obtained displacement fields).

Example 1

A plane is supported and loaded with distributed force p as shown in the figure. The material constants are E and ν.



The exact solution for this simple problem can be found

Displacement:

$$u = \frac{p}{E}x, \qquad v = -\nu\frac{p}{E}y$$

Strain:

$$\varepsilon_x = \frac{p}{E}, \qquad \varepsilon_y = -\nu\frac{p}{E}, \qquad \gamma_{xy} = 0$$

Stress:

$$\sigma_x = p, \qquad \sigma_y = 0, \qquad \tau_{xy} = 0$$

A General Formula for the Stiffness Matrix

Displacements (u, v) in a plane element are interpolated from nodal displacements (u_i, v_i) using shape functions N_i as follows,

$$\left\{ \begin{array}{c} u \\ v \end{array} \right\} = \left[\begin{array}{ccccc} N_1 & 0 & N_2 & 0 & \dots \\ 0 & N_1 & 0 & N_2 & \dots \end{array} \right] \left\{ \begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \end{array} \right\} \qquad \text{or} \qquad \mathbf{u} = \mathbf{N}\mathbf{d} \qquad (3)$$

where $\mathbf{N}$ is the *shape function matrix*, $\mathbf{u}$ the displacement vector and $\mathbf{d}$ the *nodal* displacement vector. Here we have assumed that u depends on the nodal values of u only, and v on nodal values of v only.
From strain-displacement relation $\boldsymbol{\varepsilon} = \mathbf{D}\mathbf{u}$, the strain vector is,

$$\boldsymbol{\varepsilon} = \mathbf{D}\mathbf{u} = \mathbf{D}\mathbf{N}\mathbf{d}, \qquad \text{or} \qquad \boldsymbol{\varepsilon} = \mathbf{B}\mathbf{d} \qquad (4)$$

where $\mathbf{B} = \mathbf{D}\mathbf{N}$ is the *strain-displacement matrix*.

A General Formula for the Stiffness Matrix

Consider the strain energy stored in an element,

$$\begin{aligned} U &= \frac{1}{2}\int_V \boldsymbol{\sigma}^T \boldsymbol{\varepsilon} \mathrm{d}V = \frac{1}{2}\int_V \left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \tau_{xy}\gamma_{xy}\right)\mathrm{d}V = \\ &= \frac{1}{2}\int_V \left(\mathbf{E}\boldsymbol{\varepsilon}\right)^T \boldsymbol{\varepsilon}\mathrm{d}V = \frac{1}{2}\int_V \boldsymbol{\varepsilon}^T \mathbf{E}\boldsymbol{\varepsilon}\mathrm{d}V = \\ &= \frac{1}{2}\mathbf{d}^T \int_V \mathbf{B}^T\mathbf{E}\mathbf{B}\mathrm{d}V\,\mathbf{d} = \frac{1}{2}\mathbf{d}^T\mathbf{k}\,\mathbf{d} \end{aligned}$$

From this, we obtain the general formula for the *element stiffness matrix*,

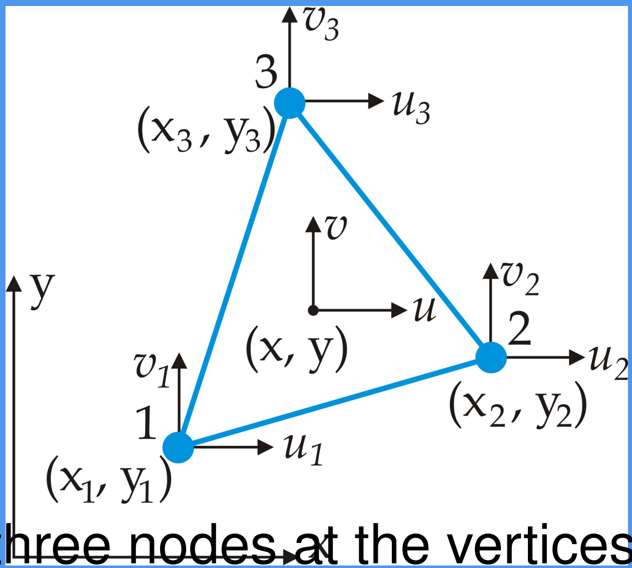
$$\mathbf{k} = \int_V \mathbf{B}^T\mathbf{E}\mathbf{B}\,\mathrm{d}V\,. \qquad (5)$$

The stiffness matrix $\mathbf{k}$ defined by (5) is symmetric since $\mathbf{E}$ is symmetric. Also note that given the material property, the behavior of $\mathbf{k}$ depends on the $\mathbf{B}$ matrix only, which in turn on shape functions. Thus, the quality of finite elements in representing the behavior of a structure is entirely determined by the choice of shape functions.

Most commonly employed 2-D elements are linear or quadratic triangles and quadrilaterals.

Constant Strain Triangle (CST or T3)

This is the simple 2-D element, which is also called ***linear triangle element***.



For this element, we have three nodes at the vertices of the triangle, which are numbered around the element in the counterclockwise direction. Each node has two degrees of freedom (can move in the x and y directions). The displacements u and v are assumed to be linear functions within the element, that is,

$$u = b_1 + b_2 x + b_3 y, \qquad v = b_4 + b_5 x + b_6 y \tag{6}$$

where $b_i \, (i = 1, 2, \ldots, 6)$ are constants. From these, the strains are found to be,

$$\varepsilon_x = b_2, \qquad \varepsilon_y = b_6, \qquad \gamma_{xy} = b_3 + b_5$$

which are constant throughout the element. Thus, we have the name ”constant strain triangle” (CST).

Constant Strain Triangle (CST or T3)

Displacements given by (**??**) should satisfy the following six equations,

$$\begin{aligned} u_1 &= b_1 + b_2x_1 + b_3y_1 \\ u_2 &= b_1 + b_2x_2 + b_3y_2 \\ &\vdots \\ v_3 &= b_4 + b_5x_3 + b_6y_3 \end{aligned}$$

Solving these equations, we can find the coefficients $b_1, b_2, \ldots,$ and b_6 in terms of nodal displacements and coordinates. Substituting these coefficients into (6) and rearranging the terms, we obtain,

$$\left\{ \begin{array}{c} u \\ v \end{array} \right\} = \left[\begin{array}{cccccc} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{array} \right] \left\{ \begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{array} \right\} \tag{7}$$

where the shape functions (linear functions in x and y) are

Constant Strain Triangle (CST or T3)

$$
\begin{aligned}
N_1 &= \frac{1}{2A}\{(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y\} \\
N_2 &= \frac{1}{2A}\{(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y\} \\
N_3 &= \frac{1}{2A}\{(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y\}
\end{aligned} \tag{8}
$$

and

$$
A = \frac{1}{2}\det\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \tag{9}
$$

is the area of the triangle (Prove this!).
Using the strain-displacement relation ($\boldsymbol{\varepsilon} = \mathbf{Du}$), results (7) and (9), we have,

$$
\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \mathbf{Bd} = \frac{1}{2A}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \tag{10}
$$

Constant Strain Triangle (CST or T3)

where $x_{ij} = x_i - x_j$ and $y_{ij} = y_i - y_j \, (i, j = 1, 2, 3)$. Again, we see constant strains within the element. From stress-strain relation (Eq.($\boldsymbol{\sigma} = \mathbf{E}\boldsymbol{\varepsilon} + \boldsymbol{\sigma}_0$), for example), we see that stresses obtained using the CST element are also constant.

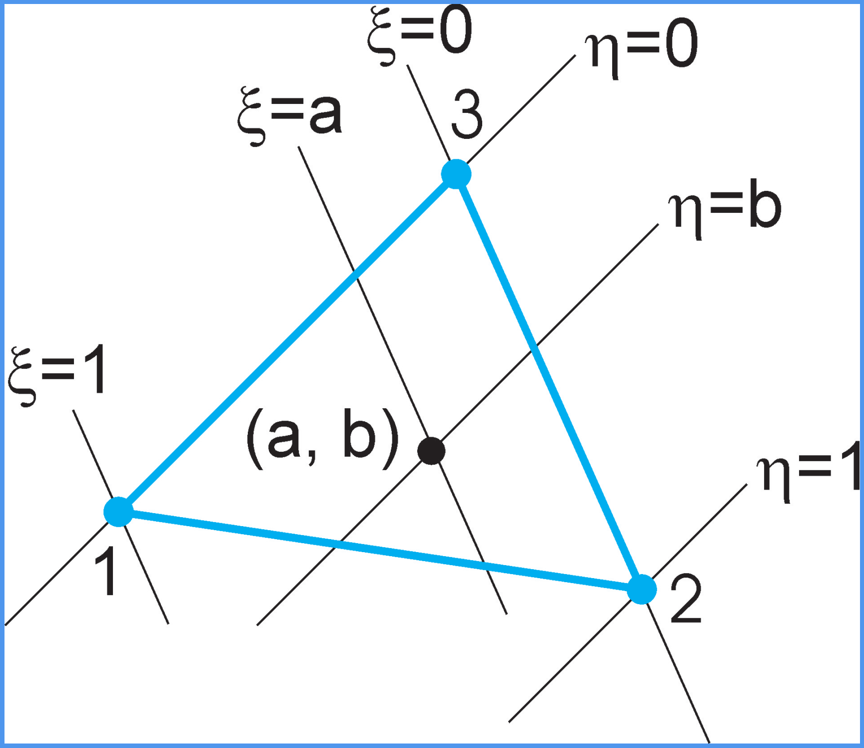
Applying formula (5), we obtain the element stiffness matrix for the CST element,

$$\mathbf{k} = \int_V \mathbf{B}^T \mathbf{E} \mathbf{B} \mathrm{d}V = tA(\mathbf{B}^T \mathbf{E} \mathbf{B}) \tag{11}$$

in which t is the thickness of the element. Notice that $\mathbf{k}$ for CST is a 6 by 6 symmetric matrix. The matrix multiplication in (11) can be carried out by a computer program.

Both the expressions of shape function in (9) and their derivations are lengthy and offer little insight into the behavior of the element.

The Natural Coordinates



We introduction the *natural coordinates* (ξ, η) on the triangle, then *the shape functions* can be represented simply by,

$$N_{,} = \xi, \qquad N_2 = \eta, \qquad N_3 = 1 - \xi - \eta \tag{12}$$

Notice that,

$$N_1 + N_2 + N_3 = 0$$

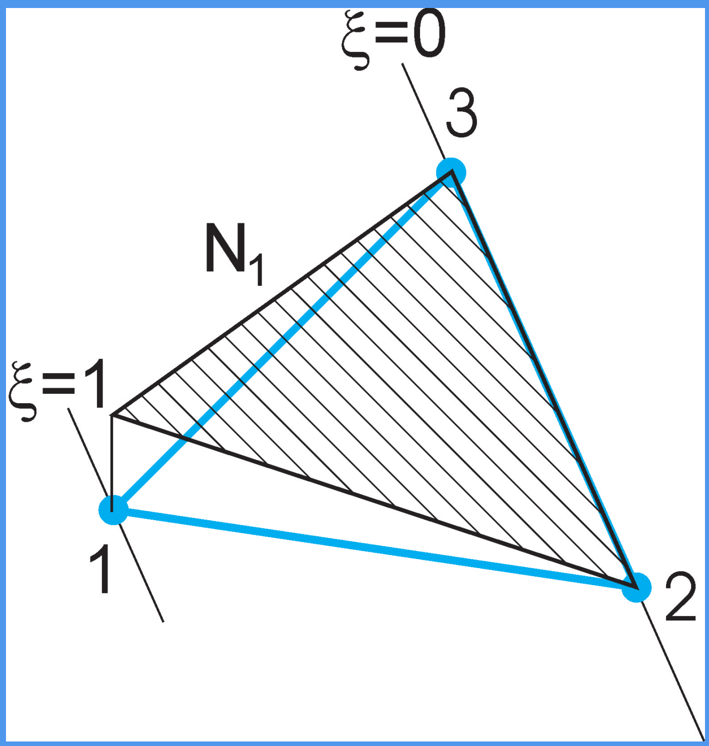
which ensures that rigid body translation is represented by the chosen shape functions.

Shape function N_1 for CTS

Also, as the 1-D case

$$N_1 = \begin{cases} 1 & \text{at node i;} \\ 0, & \text{at the other nodes} \end{cases}$$

and varies linearly within the element. The plot for shape function N_1 is shown in following figure. N_2 and N_3 have similar features.



We have two coordinate system for the element: the global coordinates $(x,\, y)$ and natural coordinates $(\xi,\, \eta)$. The relation between the two is given by

$$\begin{aligned} x &= N_1 x_1 + N_2 x_2 + N_3 x_3 \\ y &= N_1 y_1 + N_2 y_2 + N_3 y_3 \end{aligned} \qquad (13)$$

or,

$$\begin{aligned} x &= x_{13}\xi + x_{23}\eta + x_3 \\ y &= y_{13}\eta + y_{23}\eta + y_3 \end{aligned} \qquad (14)$$

where $x_{ij} = x_i - x_j$ and $y_{ij} = y_i - y_j$ $(i,\, j = 1,\, 2,\, 3)$ as defined earlier. Displacement u and v on the element can be viewed as functions of $(x,\, y)$ or $(\xi,\, \eta)$. Using the chain rule for derivatives, we have,

$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} \qquad (15)$$

where $\mathbf{J}$ is called the *Jacobian matrix* of transformation.

From (**??**), we calculate,

$$\mathbf{J} = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix}, \qquad \mathbf{J}^{-1} = \frac{1}{2A} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \tag{16}$$

where $\det \mathbf{J} = x_{13}y_{23} - x_{23}y_{13} = 2A$ has been used (A is the area of the triangular element. Prove this!).
From (15), (16), (7) and (12) we have,

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \begin{Bmatrix} u_1 - u_3 \\ u_2 - u_3 \end{Bmatrix} \tag{17}$$

Similarly,

$$\begin{Bmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \begin{Bmatrix} v_1 - v_3 \\ v_2 - v_3 \end{Bmatrix} \tag{18}$$

Using the results in (17) and (18), and the relations $\varepsilon = \mathbf{Du} = \mathbf{DNd} = \mathbf{Bd}$, we obtain the strain-displacement matrix,

$$\mathbf{B} = \frac{1}{2A}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{31} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \tag{19}$$

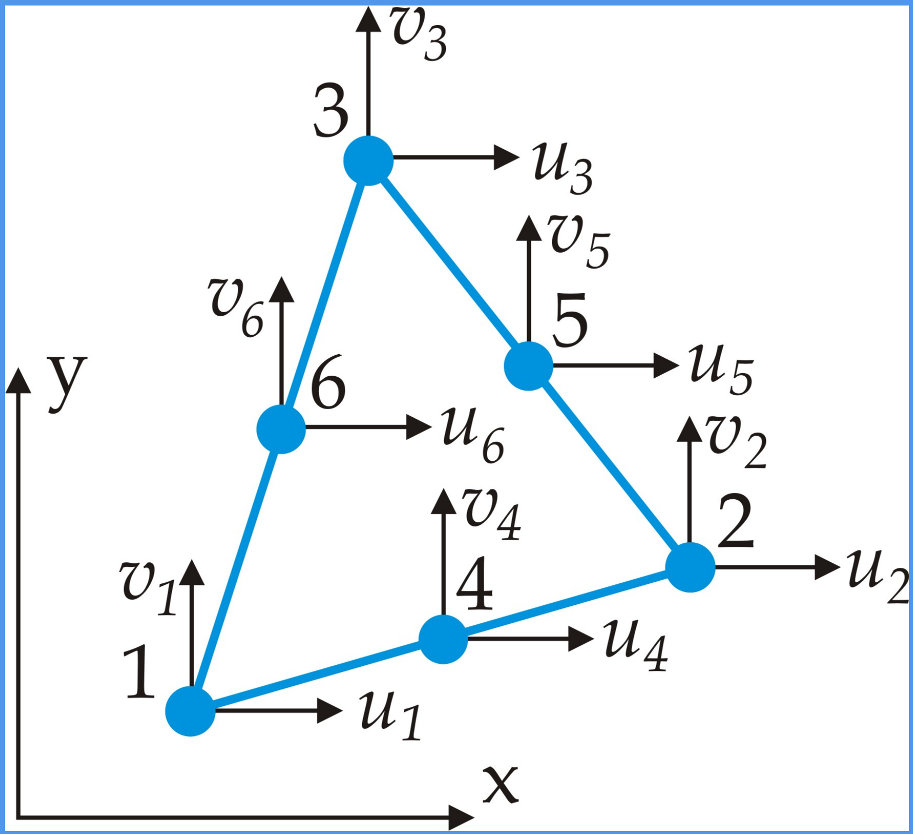
which is the same as we derived earlier in (**??**).

Applications of the CST Element:

- Use in areas where the strain gradient is small.
- Use in mesh transition areas (fine mesh to coarse mesh).
- Avoid using CST in stress concentration or other crucial areas in the structure, such as edges of holes and corners.
- Recommended for quick and preliminary FE analysis of 2-D problems.

Linear Strain Triangle (LST or T6)

This element is also called **quadratic triangular element**.



There are *six nodes* on this element: three corner nodes and three midside nodes. Each node has two degrees of freedom (DOF) as before. The displacements (u, v) are assumed to be quadratic functions of (x, y),

Linear Strain Triangle (LST or T6)

$$
\begin{aligned}
u &= b_1 + b_2 x + b_3 y + b_4 x^2 + b_5 xy + b_6 y^2 \\
u &= b_1 + b_2 x + b_3 y + b_4 x^2 + b_5 xy + b_6 y^2
\end{aligned} \qquad (20)
$$

where $b_i \ (i = 1\, 2, \ldots, 12)$ are constants. From these, the strains are found to be,

$$
\begin{aligned}
\varepsilon_x &= b_2 + 2b_4 x + b_5 y \\
\varepsilon_y &= b_9 + b_{11} x + 2b_{12} y \\
\gamma_{xy} &= (b_3 + b_8) + (b_5 + 2b_{10})x + (2b_6 + b_{11})y
\end{aligned} \qquad (21)
$$

which are linear functions. Thus, we have the ”linear strain triangle” (LST), which provides better results than the CST.

Linear Strain Triangle (LST or T6)

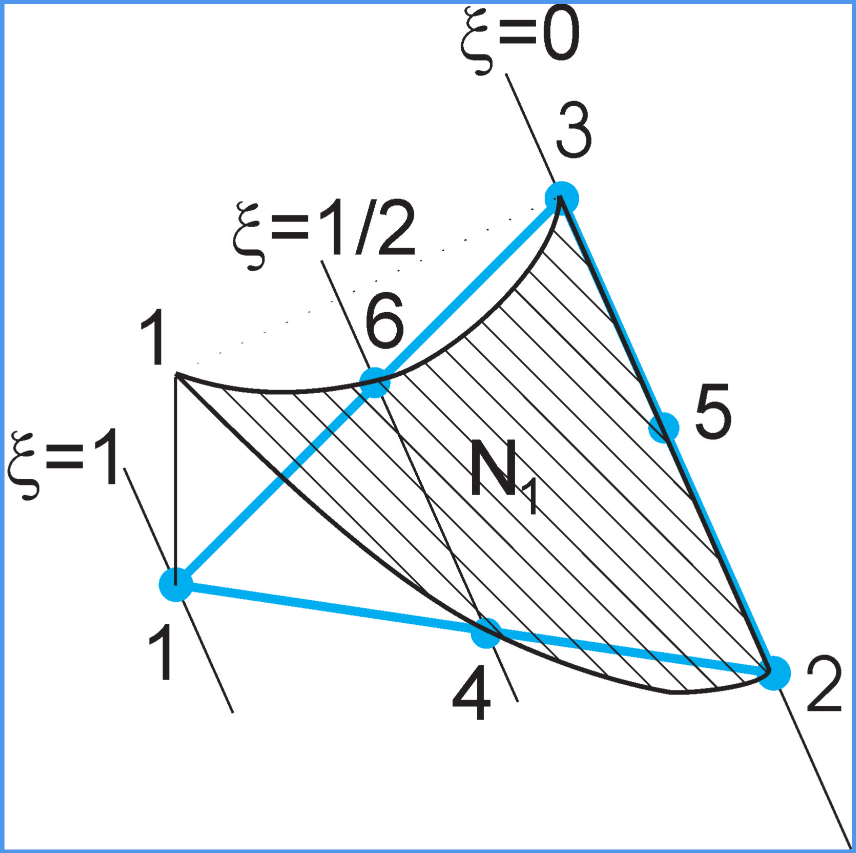
In the natural coordinates system we defined earlier, the six shape functions for the LST element are,

$$\begin{aligned} N_1 &= \xi\,(2\xi - 1) \\ N_2 &= \eta\,(2\eta - 1) \\ N_3 &= \zeta\,(2\zeta - 1) \\ N_4 &= 4\xi\eta \\ N_5 &= 4\eta\zeta \\ N_6 &= 4\zeta\xi \end{aligned} \tag{22}$$

in which $\zeta = 1 - \xi - \eta$. Each of these six shape functions represents a quadratic form on the element as shown in the figure.
Displacements can be written as,

$$u = \sum_{i=1}^{6} N_i\, u_i, \qquad v = \sum_{i=1}^{6} N_i\, v_i \tag{23}$$

Shape Function N_1 for LST

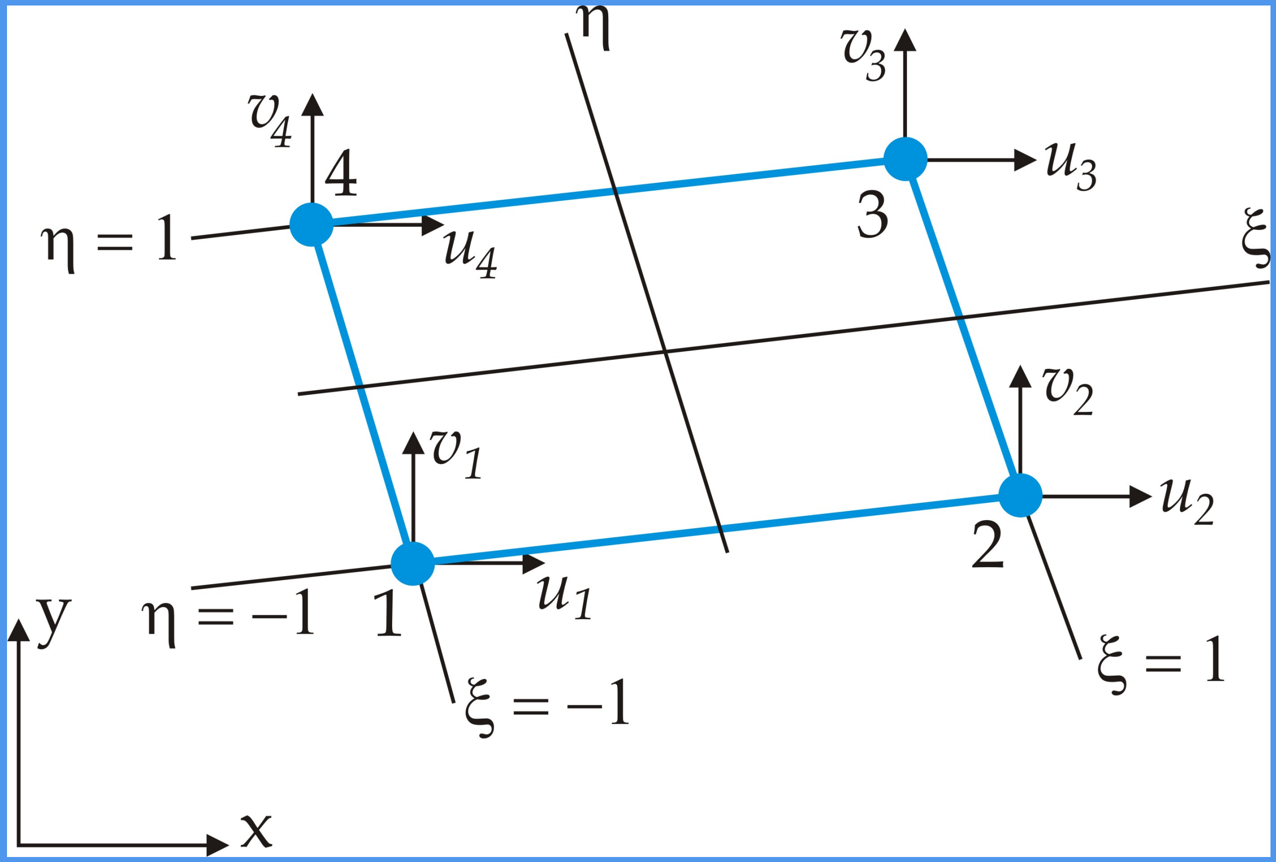


The element stiffness matrix is still given by

$$\mathbf{k} = \int_V \mathbf{B}^T \mathbf{E} \mathbf{B} \mathrm{d}V,$$

but here $\mathbf{B}^T\mathbf{E}\mathbf{B}$ is quadratic in x and y. In general, the integral has to be computed numerically.

Linear Quadrilateral Element (Q4)

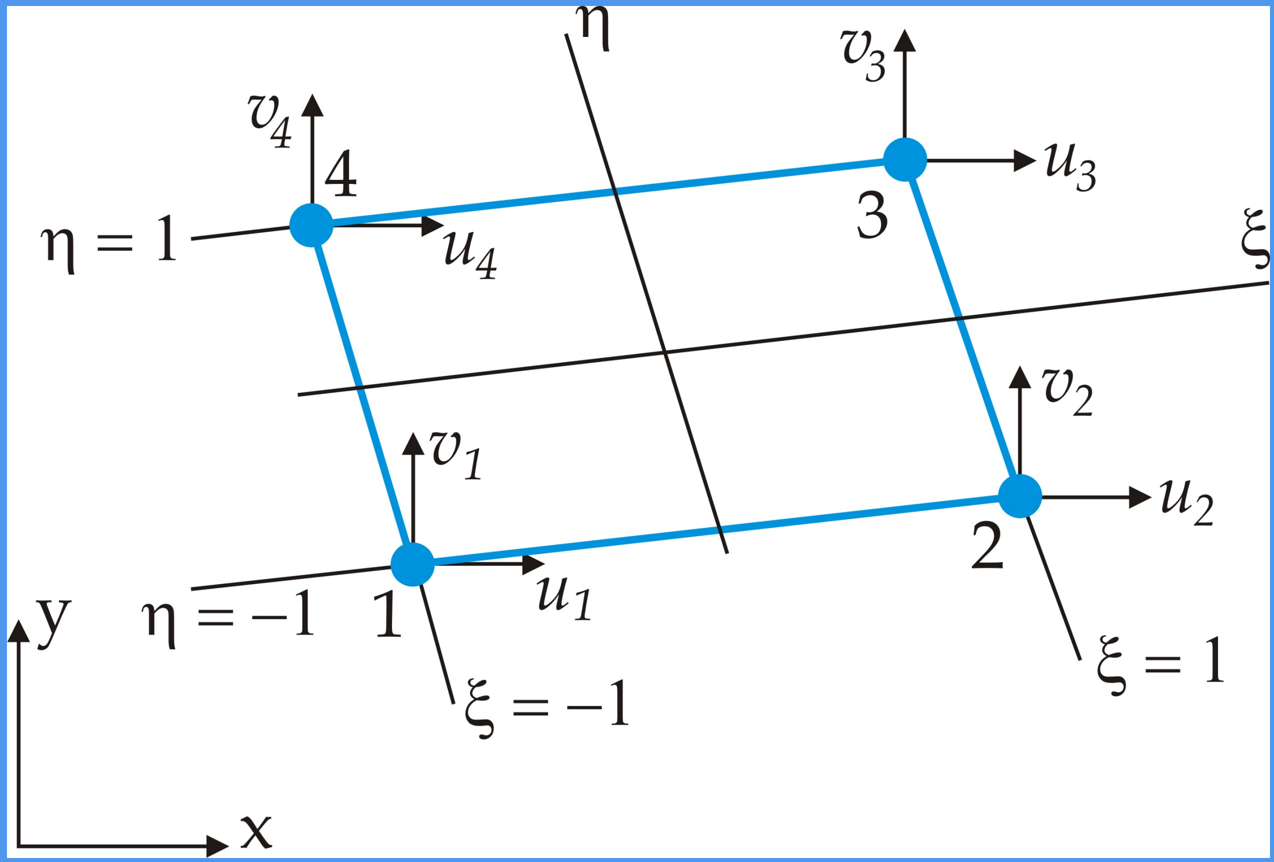


There are four nodes at the corners of the quadrilateral shape. In the natural coordinate system (ξ, η), the four shape functions are,

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta), \quad N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta), \quad N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

Linear Quadrilateral Element (Q4)



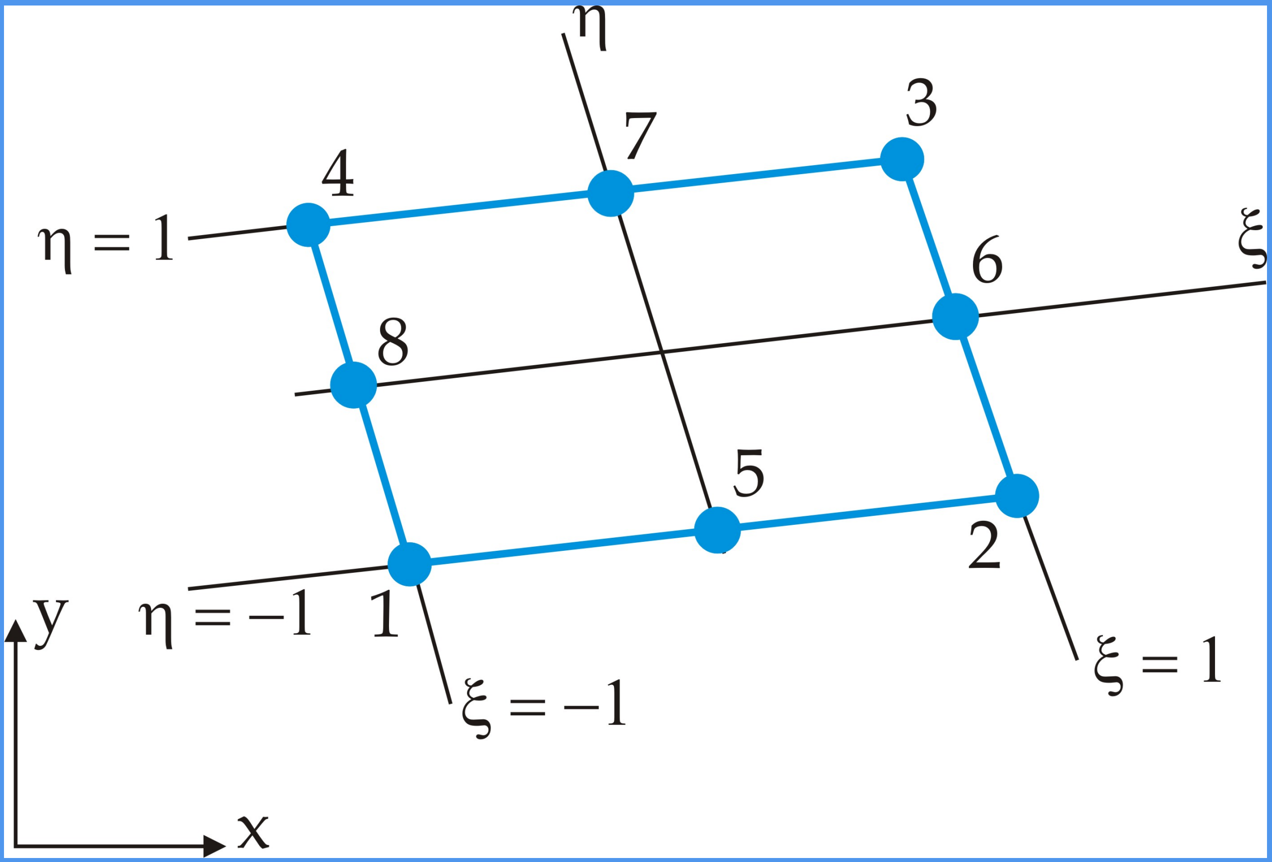
Note that $\sum_{i=1}^{4} N_i = 1$ at any point inside the element, as expected.
The displacement field is given by

$$u = \sum_{i=1}^{4} N_i\, u_i, \qquad v = \sum_{i=1}^{4} N_i\, v_i$$

which are bilinear functions over the element.

Quadratic Quadrilateral Element (Q8)

This is the most widely used element for 2-D problems due to its high accuracy in analysis and flexibility in modelling.



There are eight nodes for this element, four corners nodes and four midside nodes.

Quadratic Quadrilateral Element (Q8)

In the natural coordinate system $(\xi,\,\eta)$, the eight shape functions are,

$$N_1 = \frac{1}{4}(1-\xi)(\eta-1)(\xi+\eta+1), \qquad N_5 = \frac{1}{2}(1-\eta)(1-\xi^2)$$

$$N_2 = \frac{1}{4}(1+\xi)(\eta-1)(\eta-\xi+1), \qquad N_6 = \frac{1}{2}(1+\xi)(1-\eta^2)$$

$$N_3 = \frac{1}{4}(1+\xi)(\eta+1)(\xi+\eta-1), \qquad N_7 = \frac{1}{2}(1+\eta)(1-\xi^2)$$

$$N_4 = \frac{1}{4}(\xi-1)(\eta+1)(\xi-\eta+1), \qquad N_8 = \frac{1}{2}(1-\xi)(1-\eta^2)$$

Again, we have $\sum_{i=1}^{8} N_i = 1$ at any point inside the element.

The displacement field is given by

$$u = \sum_{i=1}^{8} N_i\,u_i, \qquad v = \sum_{i=1}^{8} N_i\,v_i$$

which are quadratic functions over the element. Strains and stresses over a quadratic quadrilateral element are linear functions, which are better representations.

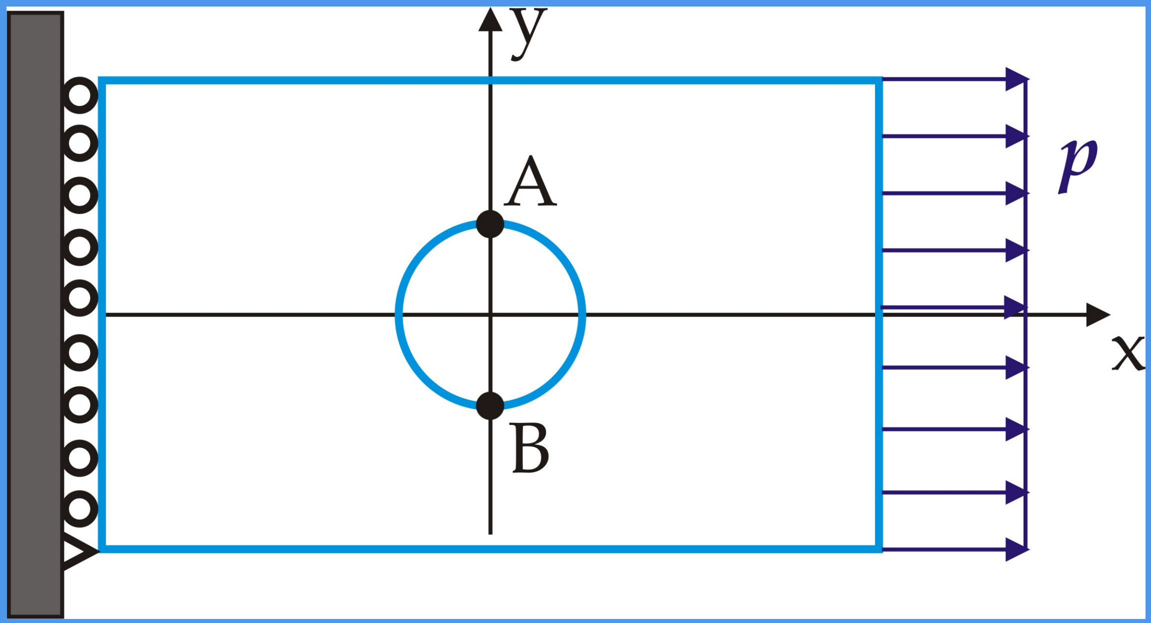
Two-Dimensional Problems

Notes:

- Q4 and T3 are usually used together in a mesh with linear elements.
- Q8 and T6 are usually applied in a mesh composed of quadratic elements.
- Quadratic elements are preferred for stress analysis, because of their high accuracy and the flexibility in modelling complex geometry, such as curved boundaries.

Example 2

A square plate with a hole at the center and under pressure in one direction.



The dimension of the plate is $100\,\mathrm{mm} \times 100\,\mathrm{mm}$, thickness is $1\,\mathrm{mm}$ and radius of the hole is $10\,\mathrm{mm}$. Assume $E = 2.1\,10^5\,\mathrm{MPa}$, $\nu = 0.3$ and $p = 100\,\mathrm{MPa}$. Find the maximum stress in the plate.

FE Analysis:

From the knowledge of stress concentrations, we should expect the maximum stresses occur at points A and B on the edge of the hole. Value of this stress should be around $3p\,(= 300\,\mathrm{MPa})$ which is the exact solution for an infinitely large plate with a hole.

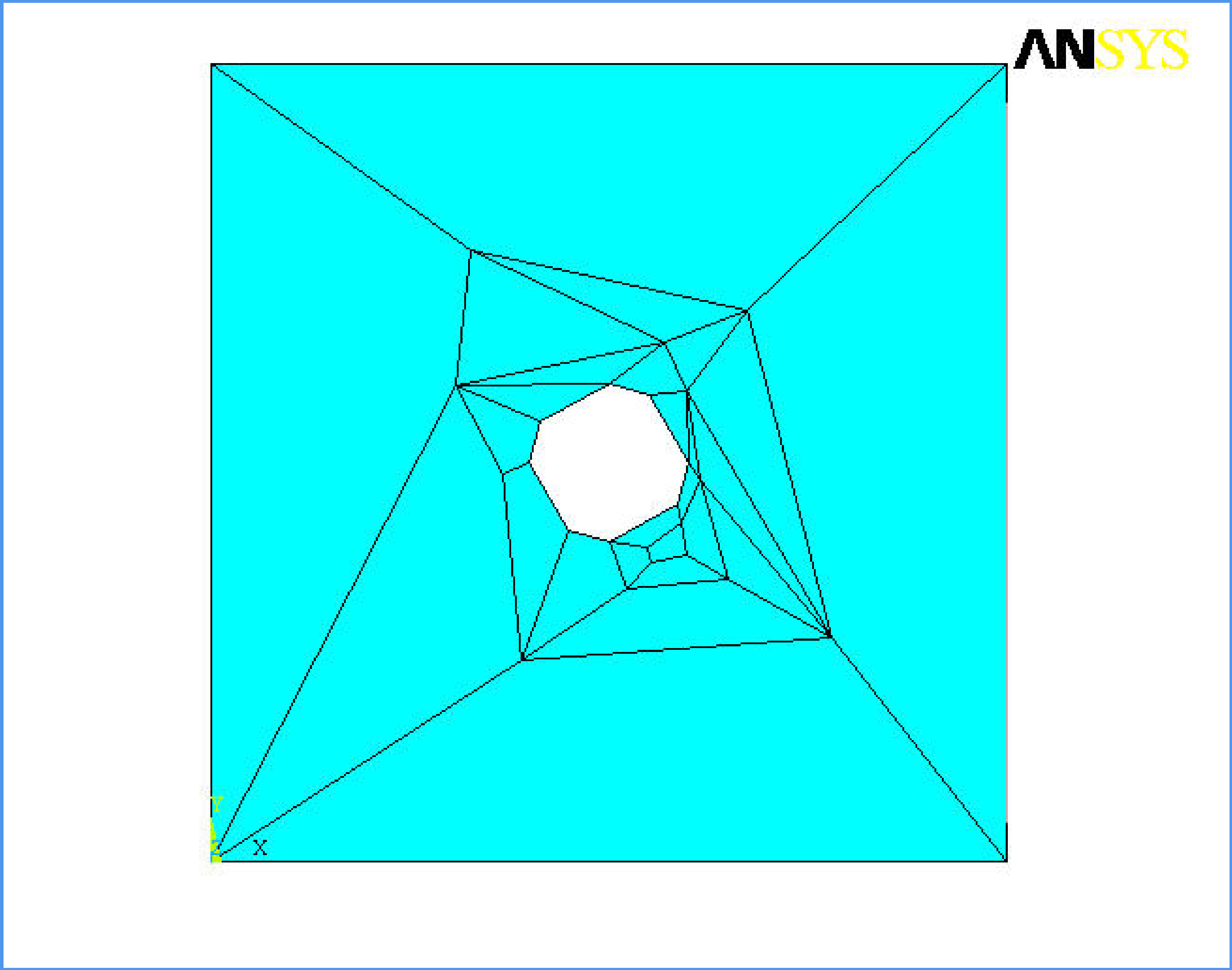
Example 2 (Result)

We use the ANSYS FEA software to do the modeling (meshing) and analysis, using quadratic triangular (T6 or LST), linear quadrilateral (Q4) and quadratic quadrilateral (Q8) elements. Linear triangles (CST or T3) is NOT available in ANSYS.

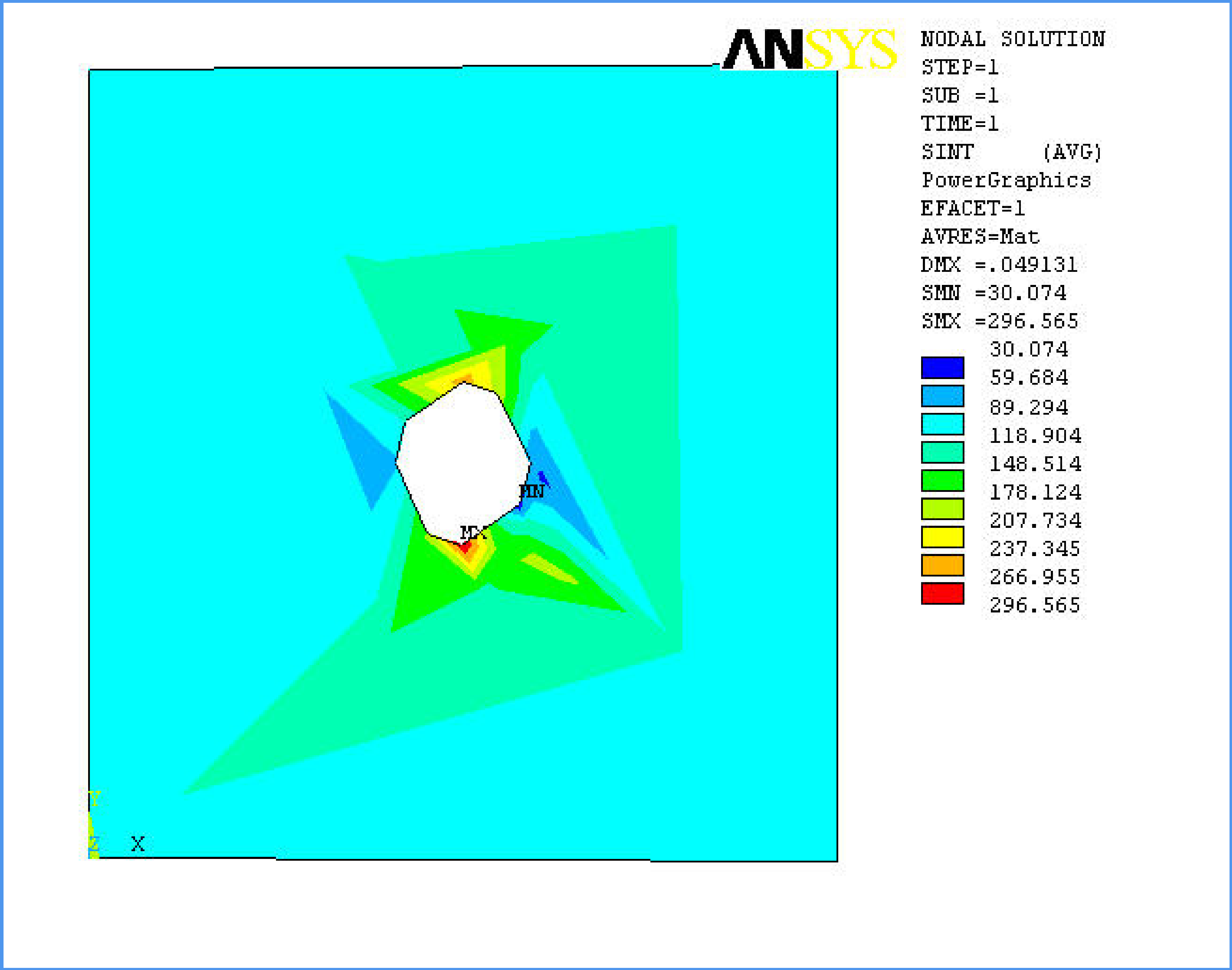
The stress calculations are listed in the following table, along with the number of elements and DOF used, for comparison.

Elem. Type	No. Elem.	No. Nodes	No. Con. Nodes	σ_{max} (MPa)	σ_{int} (MPa)	σ_{eqv} (MPa)
T6	28	68	6	243.893	243.893	227.667
T6	112	256	22	307.410	307.450	297.075
T6	1696	3516	102	323.862	323.862	321.905
Q4	26	27	4	172.021	172.021	162.552
Q4	136	162	16	289.973	289.973	275.606
Q4	2262	2372	72	326.127	326.127	320.291
Q8	26	80	6	296.565	296.565	292.053
Q8	94	322	30	313.909	313.909	309.209
Q8	1280	4012	142	321.204	321.204	319.996

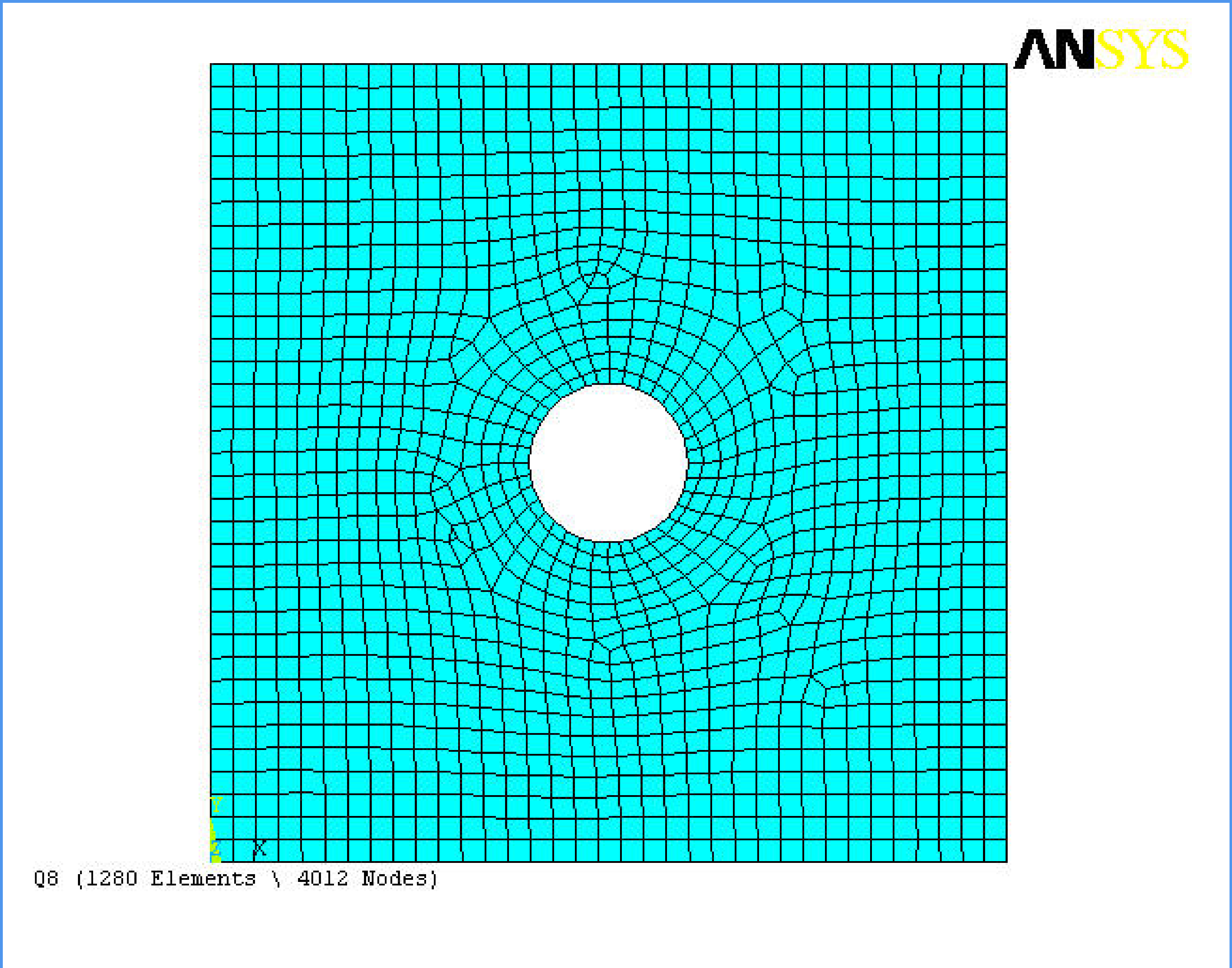
Example 2 (Result)



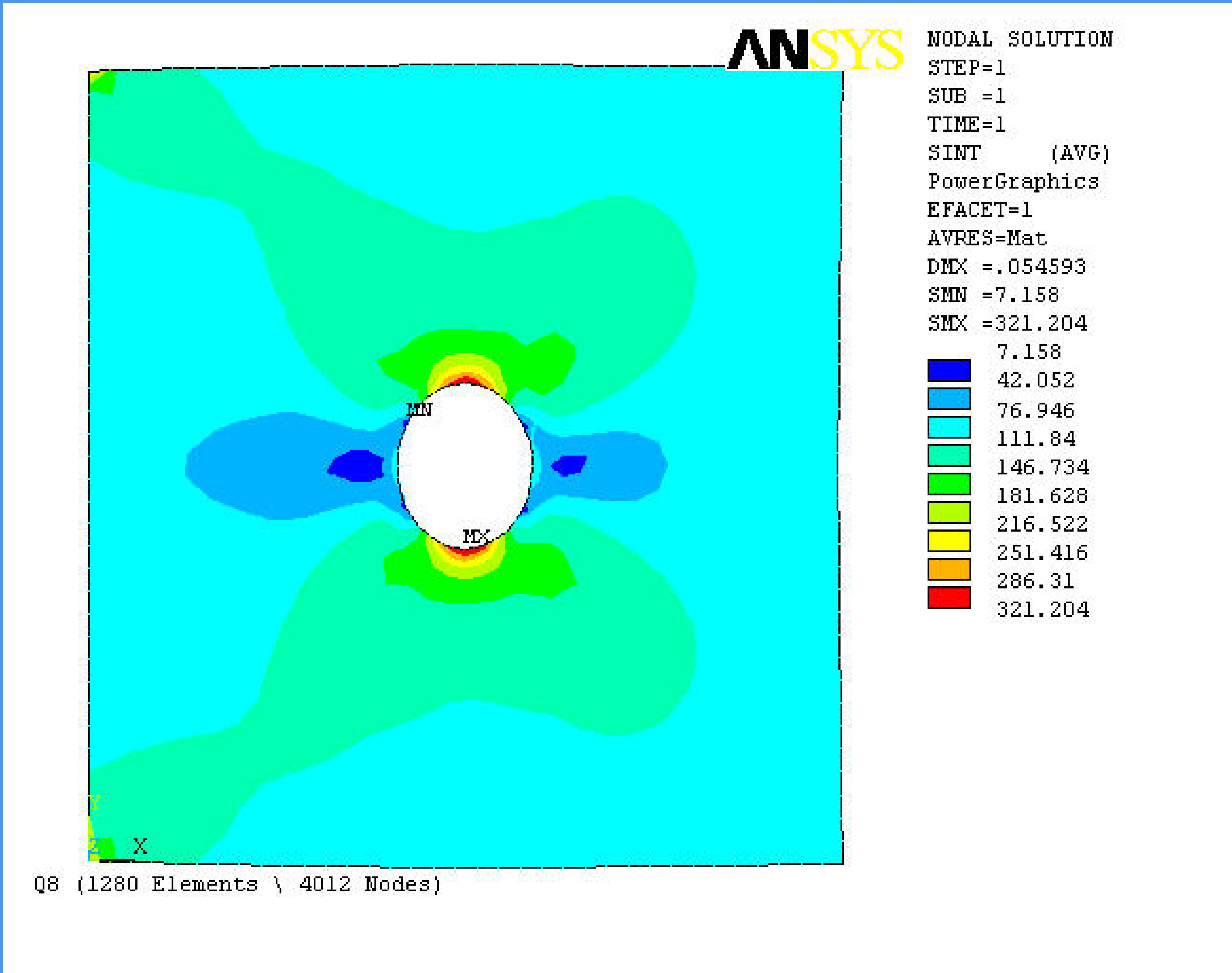
Example 2 (Result)



Example 2 (Result)

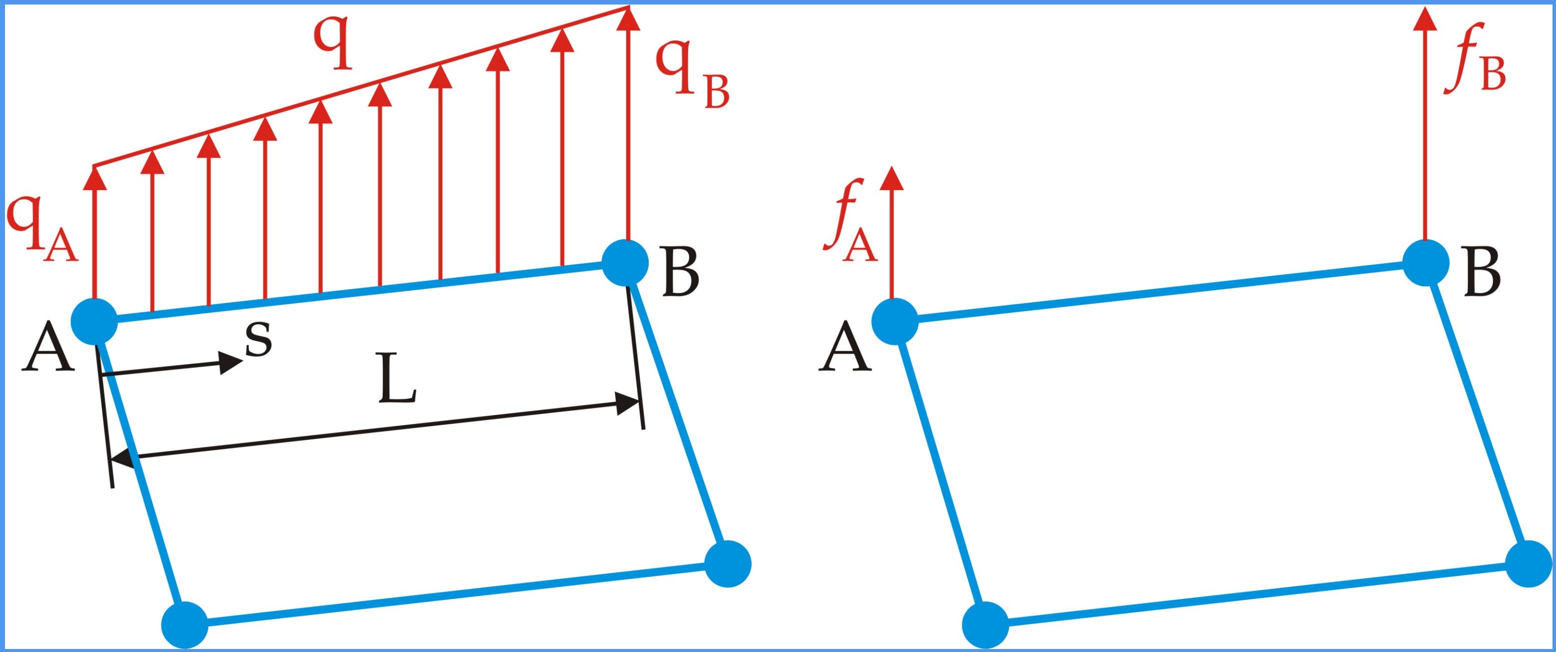


Example 2 (Result)



Transformation of Loads

Concentrated load (point forces), surface traction (pressure loads) and body force (weight) are the main types of loads applied to a surface. Both traction and body forces need to be converted to nodal forces in the FEA, since they cannot be applied to the FE model directly. The conversions of these loads are based on the same idea (the equivalent-work concept) which we have used for the cases of bar and beam elements.



Traction on a Q4 element

Suppose, for example, we have a linearly varying traction q on a Q4 element edge, as shown in the figure. The traction is normal to the boundary.

Transformation of Loads

Using the local (tangential) coordinate s, we can write the work done by traction q as,

$$W_q = t \int_0^L u_n(s) q(s) \mathrm{d}s \tag{24}$$

where t is the thickness, L the side length and u_n the component of displacement normal to the the edge AB.
For the Q4 element (linear displacement field), we have

$$u_n(s) = \left(1 - \frac{s}{L}\right) u_{nA} + \left(\frac{s}{L}\right) u_{nB} \tag{25}$$

The traction $q(s)$, which is also linear, is given in a similar way,

$$q(s) = \left(1 - \frac{s}{L}\right) q_A + \left(\frac{s}{L}\right) q_B \tag{26}$$

Transformation of Loads

Thus, we have,

$$
\begin{aligned}
W_q &= t\int_0^L \left(\begin{bmatrix} u_{nA} & u_{nB} \end{bmatrix} \begin{bmatrix} 1-\dfrac{s}{L} \\ \\ \dfrac{s}{L} \end{bmatrix} \right) \left(\begin{bmatrix} 1-\dfrac{s}{L} & \dfrac{s}{L} \end{bmatrix} \begin{bmatrix} q_A \\ \\ q_B \end{bmatrix} \right) \mathrm{d}s = \\
&= \begin{bmatrix} u_{nA} & u_{nB} \end{bmatrix} t\int_0^L \begin{bmatrix} (1-s/L)^2 & (s/L)(1-s/L) \\ (s/L)(1-s/L) & (s/L)^2 \end{bmatrix} \mathrm{d}s \begin{bmatrix} q_A \\ q_B \end{bmatrix} = \\
&= \begin{bmatrix} u_{nA} & u_{nB} \end{bmatrix} \frac{tL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} q_A \\ q_B \end{bmatrix}
\end{aligned}
$$

and the equivalent nodal force vector is,

$$
\begin{Bmatrix} f_A \\ f_B \end{Bmatrix} = \frac{tL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} q_A \\ q_B \end{Bmatrix}.
$$

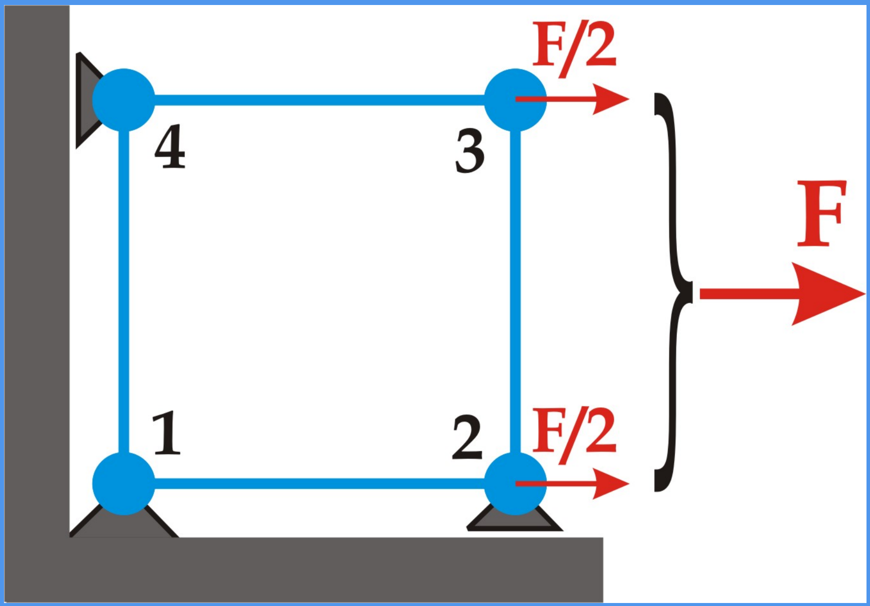
Note, for constant q, we have,

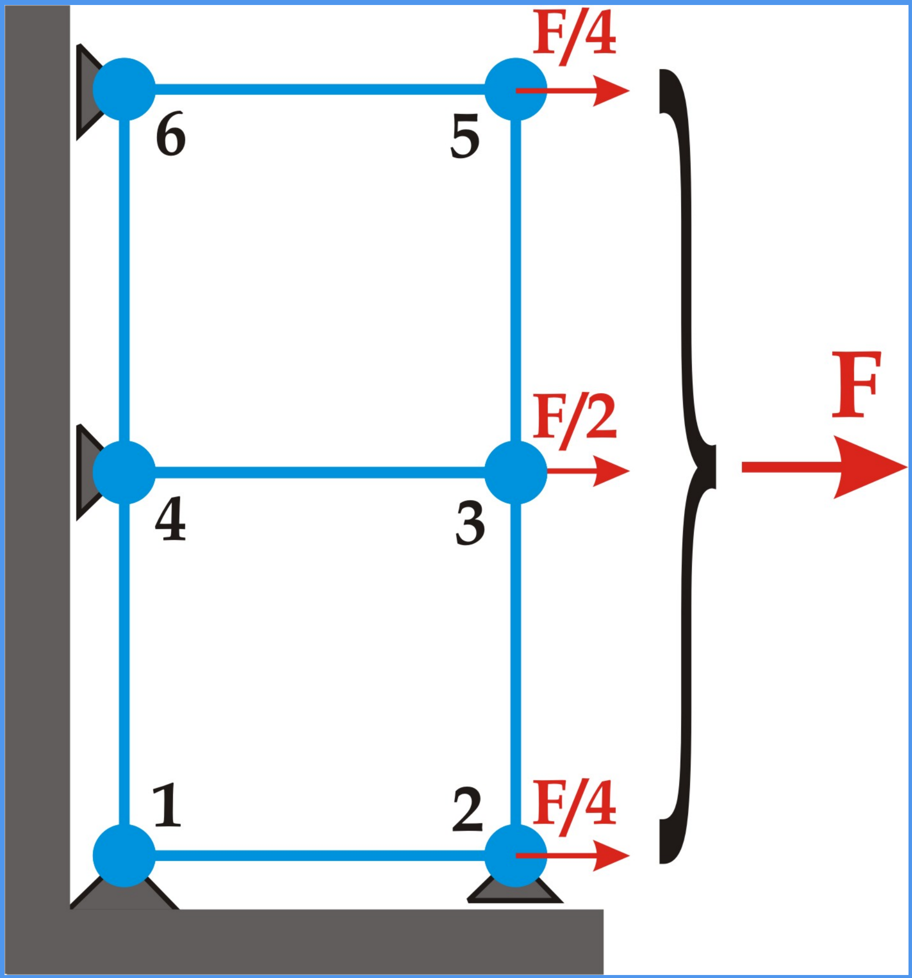
$$
\begin{Bmatrix} f_A \\ f_B \end{Bmatrix} = \frac{qtL}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.
$$

Transformation of Loads

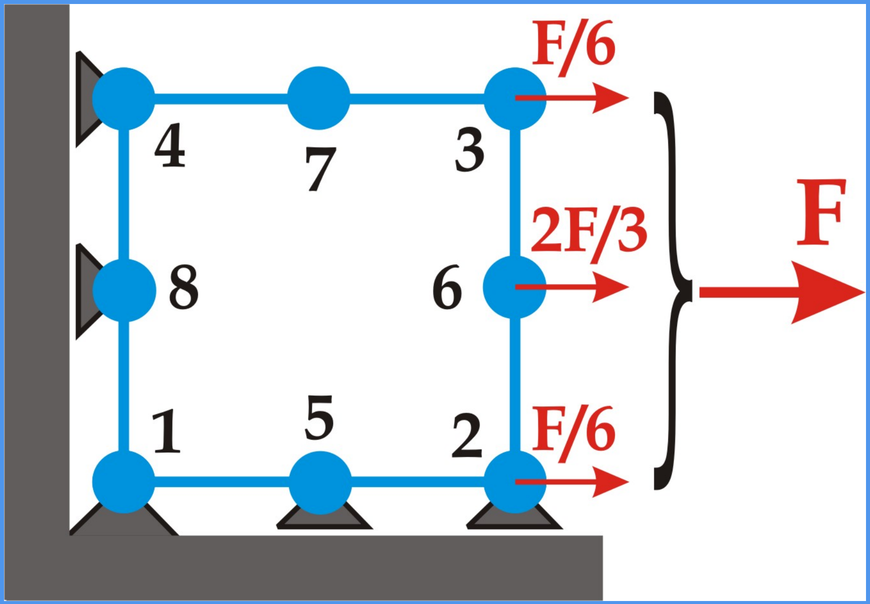
For quadratic element (either triangular or quadrilateral) the traction is converted to forces at three nodes along the edge, instead of two nodes.
Traction tangent to the boundary, as well as body forces, are converted to nodal forces in a similar way.

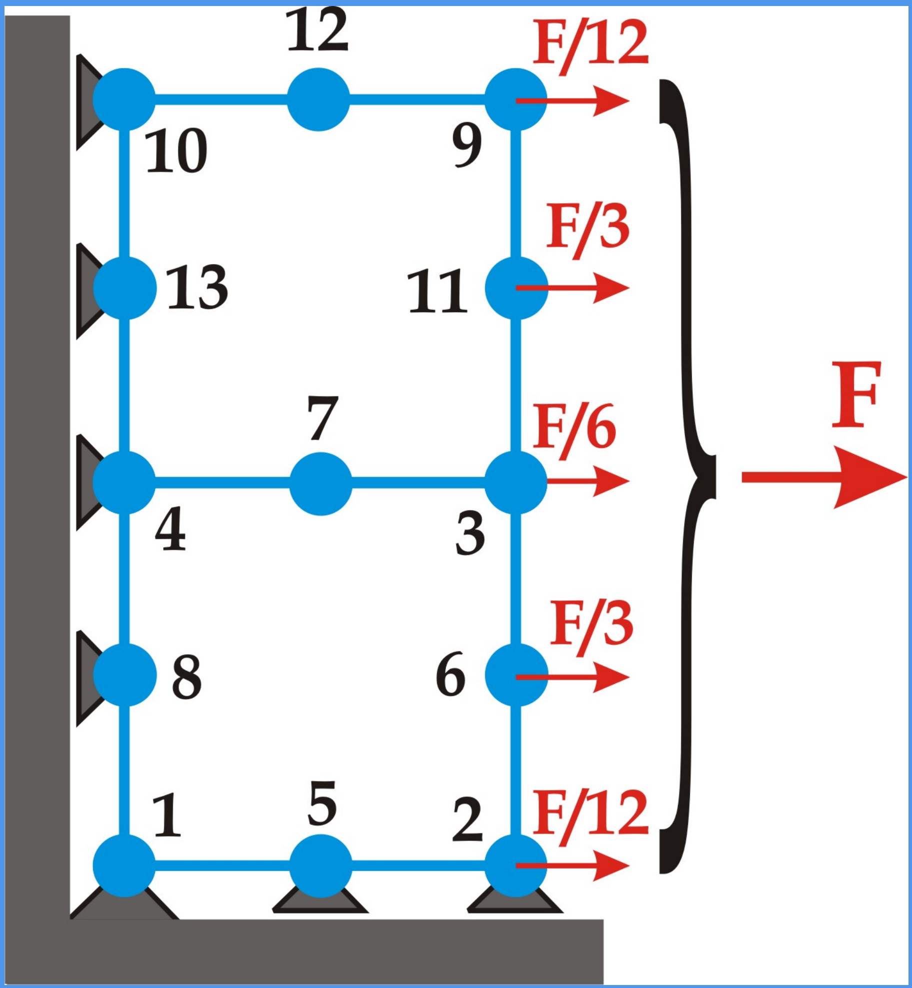
Transformation of Loads





Transformation of Loads





Stress Calculation

The stress in an element is determined by following relation,

$$\left\{ \begin{array}{c} \sigma_x \\ \sigma_y \\ \tau_x y \end{array} \right\} = \mathbf{E} \left\{ \begin{array}{c} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{array} \right\} = \mathbf{EBd} \qquad (27)$$

where $\mathbf{B}$ is the strain-nodal displacement matrix and $\mathbf{d}$ is the nodal displacement vector which is known for each element once the global FE equation has been solved.

Stresses can be evaluated at any point inside the element (such as the center) or at the nodes. Contour plots are usually used in FEA software packages (during post-process) for users to visually inspect the stress results.

Stress Calculation

The von Mises Stress:

The von Mises stress is the *effective* or *equivalent* stress for 2-D and 3-D stress analysis. For a ductile material, the stress level is considered to be safe, if

$$\sigma_e \leq \sigma_Y$$

where σ_e is the von Mises stress and σ_Y the yield stress of the material. This is a generalization of the 1-D (experimental) result to 2-D and 3-D situations.

The von Mises stress is defined by

$$\sigma_e = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \quad (28)$$

in which $\sigma_1,\ \sigma_2$ and σ_3 are the three principle stresses at the considered point in a structure.

Stress Calculation

The von Mises Stress:

For 2-D problems, the two principle stresses in the plane are determined by

$$\begin{aligned} \sigma_1^P &= \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \\ \sigma_2^P &= \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \end{aligned} \qquad (29)$$

Thus, we can also express the von Mises stress in term of stress components in the xy coordinate system. For plane stress conditions, we have,

$$\sigma_e = \sqrt{(\sigma_x + \sigma_y)^2 - 3(\sigma_x \sigma_y - \tau_{xy}^2)} \qquad (30)$$

Stress Calculation

Averaged Stresses:

Stresses are usually averaged at nodes in FEA software packages to provide more accurate stress values. This option should be turned off at nodes between two materials or other geometry discontinuity locations where stress discontinuity does exist.

Discussions

1) *Know the behaviors of each type of elements:*

T3 and Q4: linear displacement, constant strain and stress;

T6 and Q8: quadratic displacement, linear strain and stress.

2) *Choose the right type of elements for a given problem:*

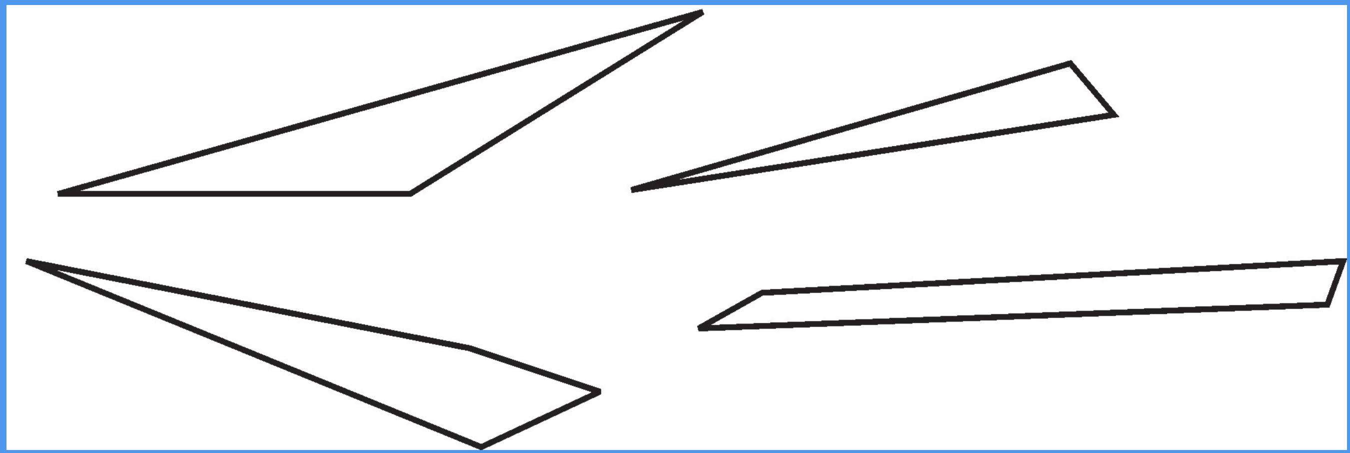
When in doubt, use higher order elements or a finer mesh.

3) *Avoid elements with large aspect ratios and corner angles:*

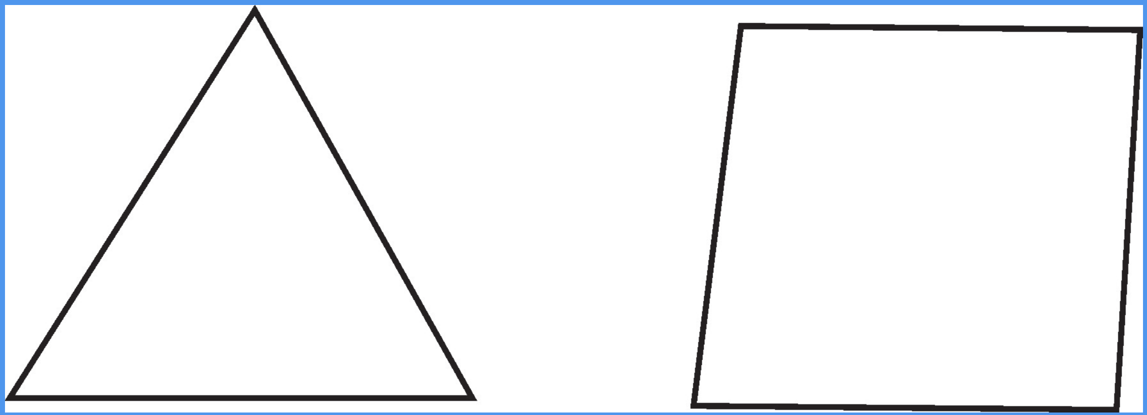
$$\text{Aspect ratio} = L_{max} / L_{min}$$

where L_{max} and L_{min} are the largest and smallest characteristic lengths of an element, respectively.

Discussions



Elements with Bad Shapes

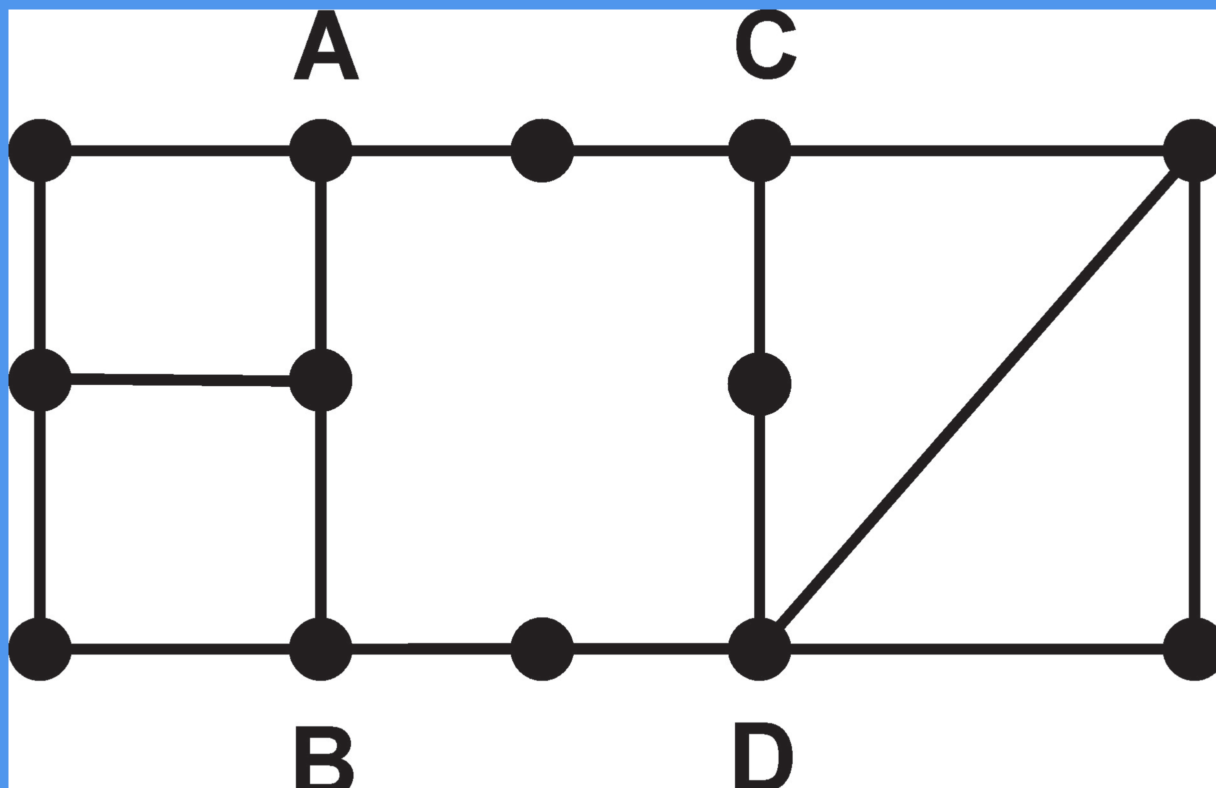


Elements with Nice Shapes

Discussions

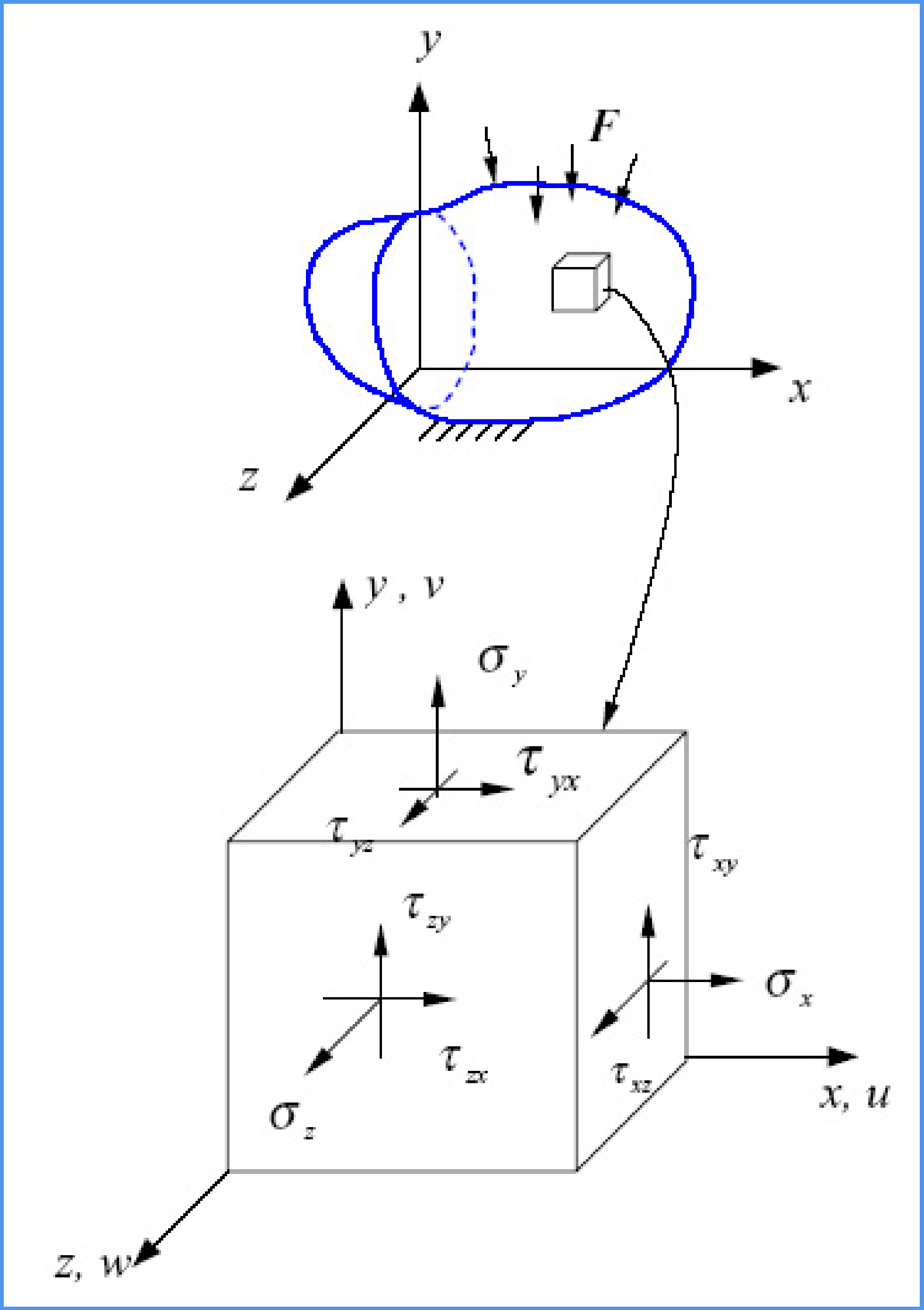
4) *Connect the elements properly:*

Don’t leave unintended gaps or free elements in FE models.



Improper connections (gaps along AB and CD)

3D Elasticity Theory



3D Elasticity Theory

Stress State:

$$\boldsymbol{\sigma} = \{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} \quad \text{or} \quad [\sigma_{ij}] \tag{31}$$

Strains:

$$\boldsymbol{\varepsilon} = \{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \quad \text{or} \quad [\varepsilon_{ij}] \tag{32}$$

3D Elasticity Theory

Stress-Strains relation:

$$\left\{\begin{array}{c}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{zx}\end{array}\right\} = \frac{\mathrm{E}}{(1+\nu)(1-2\nu)}\left[\begin{array}{cccccc}1-\nu & \nu & \nu & 0 & 0 & 0\\ \nu & 1-\nu & \nu & 0 & 0 & 0\\ \nu & \nu & 1-\nu & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2}\end{array}\right]\left\{\begin{array}{c}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{zx}\end{array}\right\} \quad (33)$$

or

$$\boldsymbol{\sigma} = \mathbf{E}\,\boldsymbol{\varepsilon}$$

Displacement:

$$\mathbf{u} = \left\{\begin{array}{c}u(x,\,y,\,z)\\ v(x,\,y,\,z)\\ w(x,\,y,\,z)\end{array}\right\} = \left\{\begin{array}{c}u_1\\ u_2\\ u_3\end{array}\right\} \quad (34)$$

3D Elasticity Theory

Strain-Displacement Relation:

$$\varepsilon_x = \frac{\partial u}{\partial x}, \qquad \varepsilon_y = \frac{\partial v}{\partial y}, \qquad \varepsilon_z = \frac{\partial w}{\partial z},$$
$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}, \qquad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \qquad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \qquad (35)$$

or

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad (i,\, j = 1,\, 2,\, 3)$$

or simply

$$\varepsilon_{ij}\,\frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \qquad \text{(tensor notation)}$$

3D Elasticity Theory

Equilibrium Equations:

$$\begin{aligned}
\frac{\partial \sigma_x}{\partial x} + \frac{\tau_{xy}}{\partial y} + \frac{\tau_{xz}}{\partial z} + f_x &= 0\,, \\
\frac{\partial \tau_{yx}}{\partial x} + \frac{\sigma_y}{\partial y} + \frac{\tau_{yz}}{\partial z} + f_y &= 0\,, \\
\frac{\partial \tau_{zx}}{\partial x} + \frac{\tau_{zy}}{\partial y} + \frac{\sigma_z}{\partial z} + f_z &= 0\,,
\end{aligned} \tag{36}$$

or

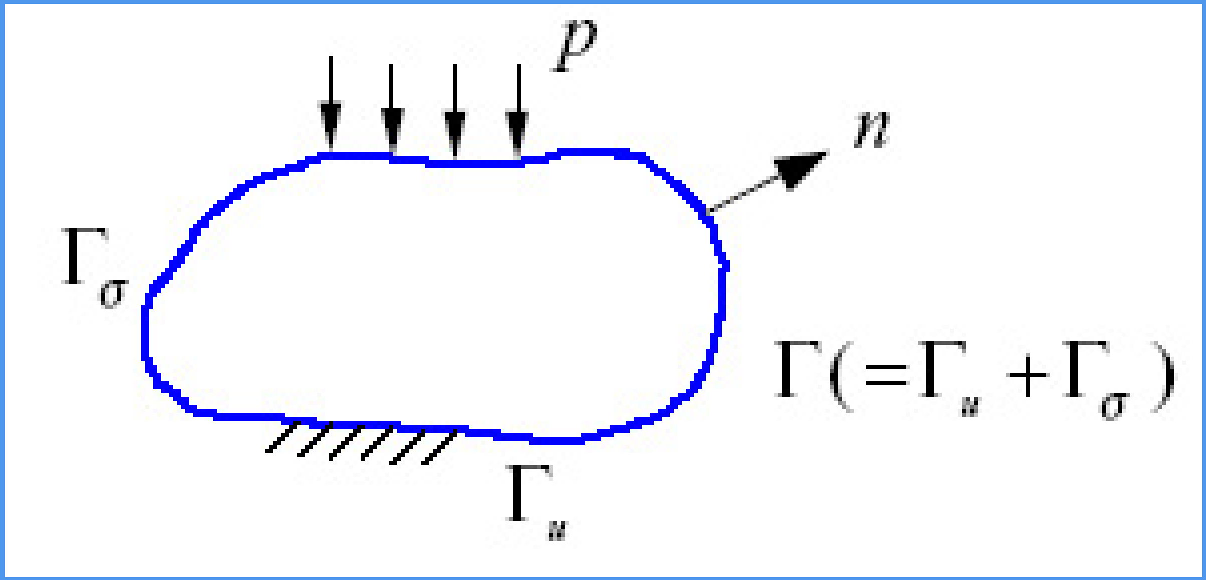
$$\sigma_{ij,j} + f_i = 0$$

Boundary Conditions (BC's):

$$\begin{aligned}
u_i &= \bar{u}_i, \qquad \text{on} \quad \Gamma_u \,\text{(specified displacement)} \\
t_i &= \bar{t}_i, \qquad \text{on} \quad \Gamma_\sigma \,\text{(specified traction)}
\end{aligned} \tag{37}$$

(traction $t_i = \sigma_{ij}\, n_j$)

3D Elasticity Theory



Stress Analysis:

Solving equations in (36) under the BC's in (37).

II. Finite Element Formulation

Displacement field:

$$\left.\begin{aligned} u &= \sum_{i=1}^{N} N_i\, u_i \\ v &= \sum_{i=1}^{N} N_i\, v_i \\ w &= \sum_{i=1}^{N} N_i\, w_i \end{aligned}\right\} \quad \text{where } u_i,\ v_i,\ w_i\ \ldots \text{ nodal values} \tag{38}$$

In matrix form:

$$\left\{\begin{array}{c} u \\ v \\ w \end{array}\right\}_{(3\times 1)} = \left[\begin{array}{ccccccc} N_1 & 0 & 0 & N_2 & 0 & 0 & \ldots \\ 0 & N_1 & 0 & 0 & N_2 & 0 & \ldots \\ 0 & 0 & N_1 & 0 & 0 & N_2 & \ldots \end{array}\right]_{(3\times 3N)} \left\{\begin{array}{c} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \\ \vdots \end{array}\right\}_{(3N\times 1)} \tag{39}$$

or $$\mathbf{u} = \mathbf{N}\,\mathbf{d}$$

II. Finite Element Formulation

Using relations (35) and (38), we can derive the strain vector

$$\boldsymbol{\varepsilon}_{(6\times1)} = \mathbf{B}_{(6\times3N)}\,\mathbf{d}_{3N\times1}$$

Stiffness Matrix:

$$\mathbf{K}_{(3\times N)} = \int_V \mathbf{B}^T_{(3N\times6)}\,\mathbf{E}_{(6\times6)}\,\mathbf{B}_{(6\times3N)}\,\mathrm{d}V \qquad (40)$$

Numerical quadratures are often needed to evaluate the above integration.

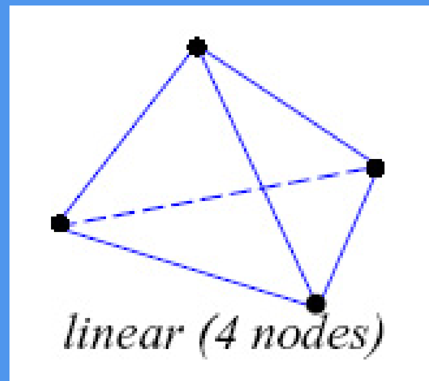
Rigid-body motions for 3-D bodies (6 components):

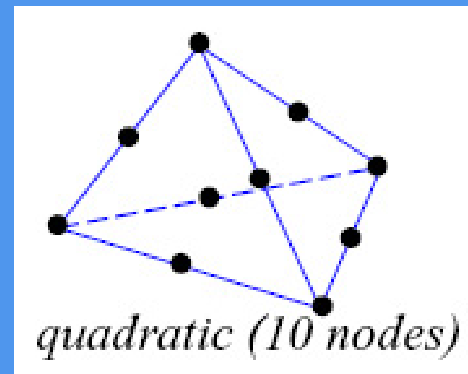
3 translations, 3 rotations.

These rigid-body motions (singularity of the system of equations) must be removed from the FEA model to ensure the quality of the analysis.

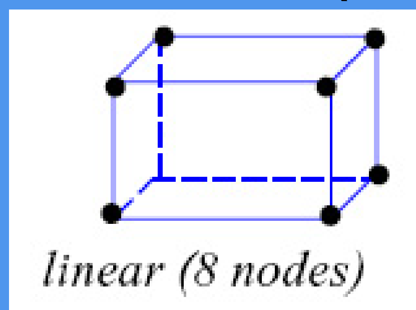
III Typical 3-D Solid Elements

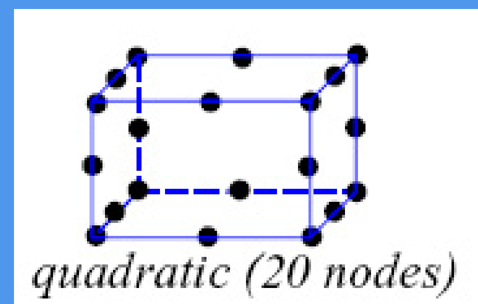
Tetrahedron:



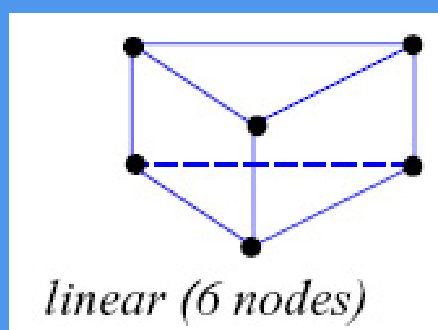


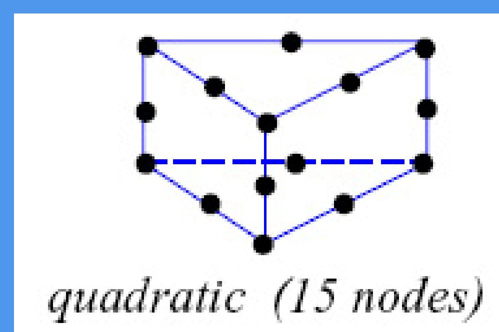
Hexahedron (brick):





Penta:

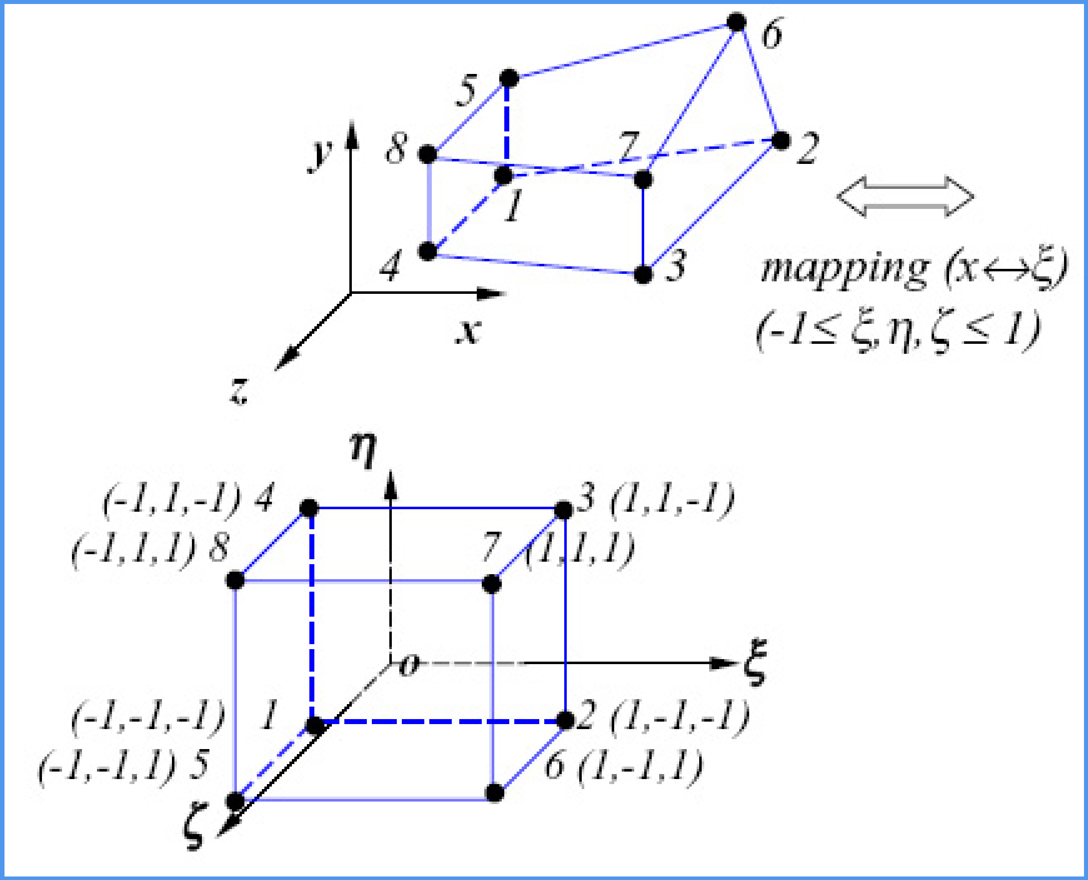




Avoid using the linear (4-node) tetrahedron element in 3-D stress analysis (Inaccurate! But it is OK for dynamic analysis).

III Typical 3-D Solid Elements

Element formulation: – Linear Hexahedron Element



Displacement field in the element:

$$u_i = \sum_{i=1}^{8} N_i\, u_i, \qquad v_i = \sum_{i=1}^{8} N_i\, v_i, \qquad w_i = \sum_{i=1}^{8} N_i\, w_i \qquad (41)$$

III Typical 3-D Solid Elements

Shape functions:

$$
\begin{aligned}
N_1(\xi,\eta,\zeta) &= \frac{1}{8}(1-\xi)(1-\eta)(1-\zeta)\,, \\
N_2(\xi,\eta,\zeta) &= \frac{1}{8}(1+\xi)(1-\eta)(1-\zeta)\,, \\
N_3(\xi,\eta,\zeta) &= \frac{1}{8}(1+\xi)(1+\eta)(1-\zeta)\,, \\
&\vdots \qquad\qquad \vdots \\
N_8(\xi,\eta,\zeta) &= \frac{1}{8}(1-\xi)(1+\eta)(1+\zeta)\,,
\end{aligned}
\tag{42}
$$

Note that we have the following relations for the shape functions:

$$
\begin{aligned}
N_i(\xi_j,\eta_j,\zeta_j) &= \delta_{ij}, \qquad i,\,j = 1,\,2,\,\ldots,\,8. \\
\sum_{i=1}^{8} N_i(\xi,\eta,\zeta) &= 1
\end{aligned}
$$

III Typical 3-D Solid Elements

Coordinate Transformation (Mapping):

$$x = \sum_{i=1}^{8} N_i \, x_i \, , \qquad y = \sum_{i=1}^{8} N_i \, y_i \, , \qquad z = \sum_{i=1}^{8} N_i \, z_i \, . \tag{43}$$

The same shape functions are used as for the displacement field.
$\Rightarrow$ **Isoparametric element.**

Jacobian Matrix:

$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\[2ex] \dfrac{\partial u}{\partial \eta} \\[2ex] \dfrac{\partial u}{\partial \zeta} \end{Bmatrix} = \underbrace{\begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} & \dfrac{\partial z}{\partial \xi} \\[2ex] \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} & \dfrac{\partial z}{\partial \eta} \\[2ex] \dfrac{\partial x}{\partial \zeta} & \dfrac{\partial y}{\partial \zeta} & \dfrac{\partial z}{\partial \zeta} \end{bmatrix}}_{\equiv \mathbf{J} \text{ Jacobian matrix}} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\[2ex] \dfrac{\partial u}{\partial y} \\[2ex] \dfrac{\partial u}{\partial z} \end{Bmatrix} \tag{44}$$

III Typical 3-D Solid Elements

$$\Rightarrow \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\[2ex] \dfrac{\partial u}{\partial y} \\[2ex] \dfrac{\partial u}{\partial z} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\[2ex] \dfrac{\partial u}{\partial \eta} \\[2ex] \dfrac{\partial u}{\partial \zeta} \end{Bmatrix}, \qquad \left(\frac{\partial u}{\partial \xi} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial \xi} u_i \,, \ \text{etc.} \right)$$

and

$$\Rightarrow \begin{Bmatrix} \dfrac{\partial v}{\partial x} \\[2ex] \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial v}{\partial z} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \dfrac{\partial v}{\partial \xi} \\[2ex] \dfrac{\partial v}{\partial \eta} \\[2ex] \dfrac{\partial v}{\partial \zeta} \end{Bmatrix},$$

also for w.

III Typical 3-D Solid Elements

$\Rightarrow$

$$\boldsymbol{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} \\ \dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z} \\ \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x} \end{Bmatrix} = \cdots use\,(15) = \mathbf{B}\,\mathbf{d}$$

where $\mathbf{d}$ is the nodal displacement vector,
i.e.,

$$\boldsymbol{\varepsilon}_{(6\times 1)} = \mathbf{B}_{(6\times 24)}\,\mathbf{d}_{24\times 1} \tag{45}$$

Strain energy,

$$U = \frac{1}{2}\int_V \boldsymbol{\sigma}^T \boldsymbol{\varepsilon}\,\mathrm{d}V = \frac{1}{2}\int_V (\mathbf{E}\boldsymbol{\varepsilon})^T \boldsymbol{\varepsilon}\,\mathrm{d}V = \frac{1}{2}\int_V \boldsymbol{\varepsilon}^T \mathbf{E}\,\boldsymbol{\varepsilon}\,\mathrm{d}V = \frac{1}{2}\mathbf{d}^T \left[\int_V \mathbf{B}^T \mathbf{E}\,\mathbf{B}\,\mathrm{d}V\right]\mathbf{d} \tag{46}$$

III Typical 3-D Solid Elements

Element stiffness matrix,

$$\mathbf{k}_{(24\times 24)} = \int_V \mathbf{B}^T{}_{(24\times 6)} \mathbf{E}_{(6\times 6)} \mathbf{B}_{6\times 24} \, \mathrm{d}V \tag{47}$$

In $\xi \, \eta \, \zeta$ coordinates:

$$\mathrm{d}V = (\det \mathbf{J}) \, \mathrm{d}\xi \, \mathrm{d}\eta \, \mathrm{d}\zeta \tag{48}$$

$$\Rightarrow \qquad \mathbf{k} = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \mathbf{B}^T \mathbf{E} \, \mathbf{B} \, (\det \mathbf{J}) \, \mathrm{d}\xi \, \mathrm{d}\eta \, \mathrm{d}\zeta \tag{49}$$

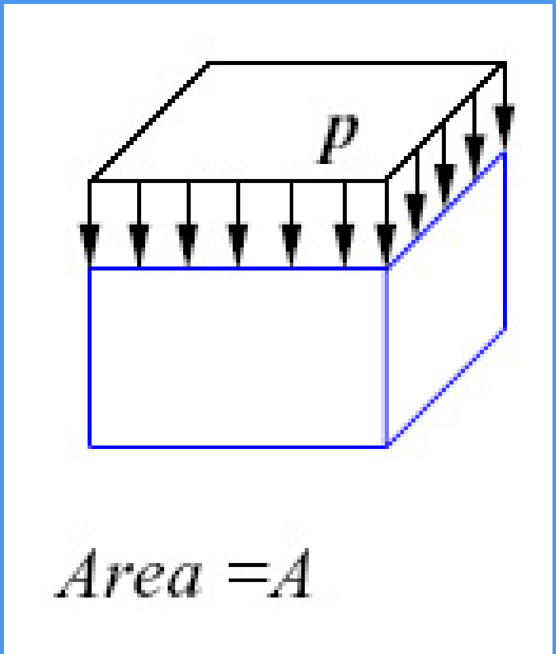
(Numerical integration)

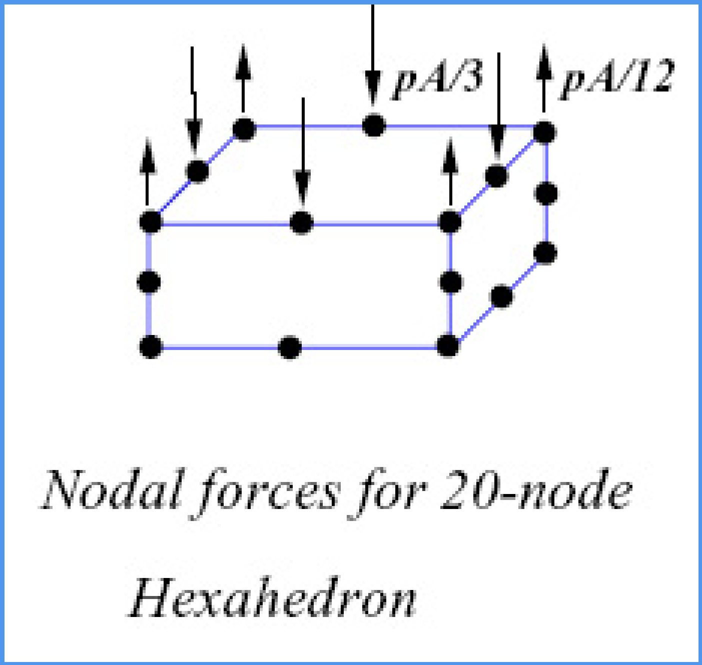
3-D elements usually do not use rotational DOFs.

III Typical 3-D Solid Elements

Loads:

Distributed loads ⇒ Nodal forces





Stresses:

$$\boldsymbol{\sigma} = \mathbf{E}\boldsymbol{\varepsilon} = \mathbf{E}\mathbf{B}\mathbf{d}$$

Principal stresses:

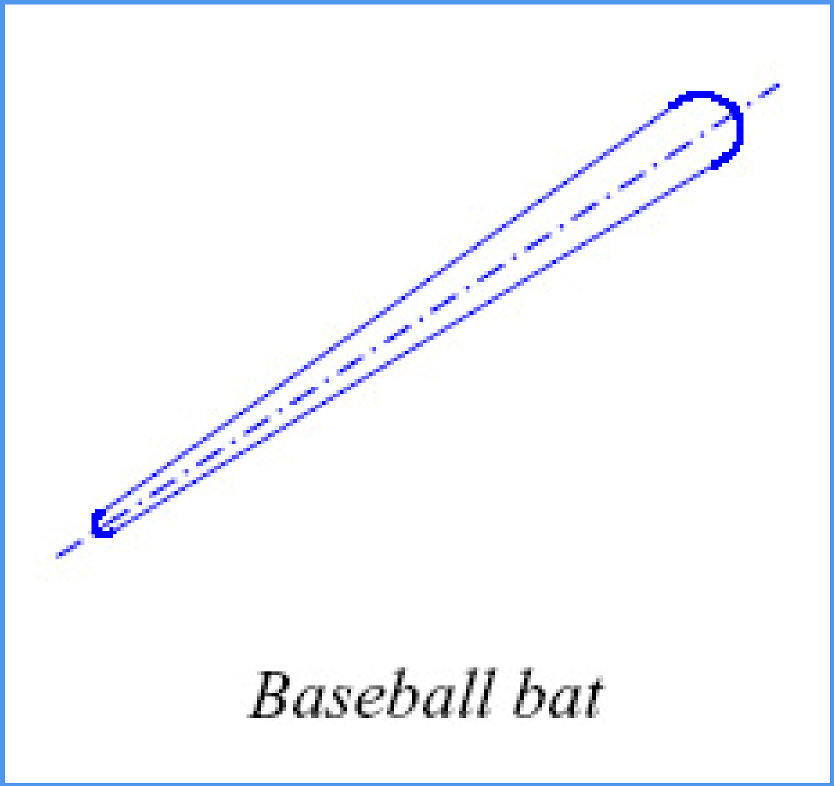
$$\sigma_1,\ \sigma_2,\ \sigma_3.$$

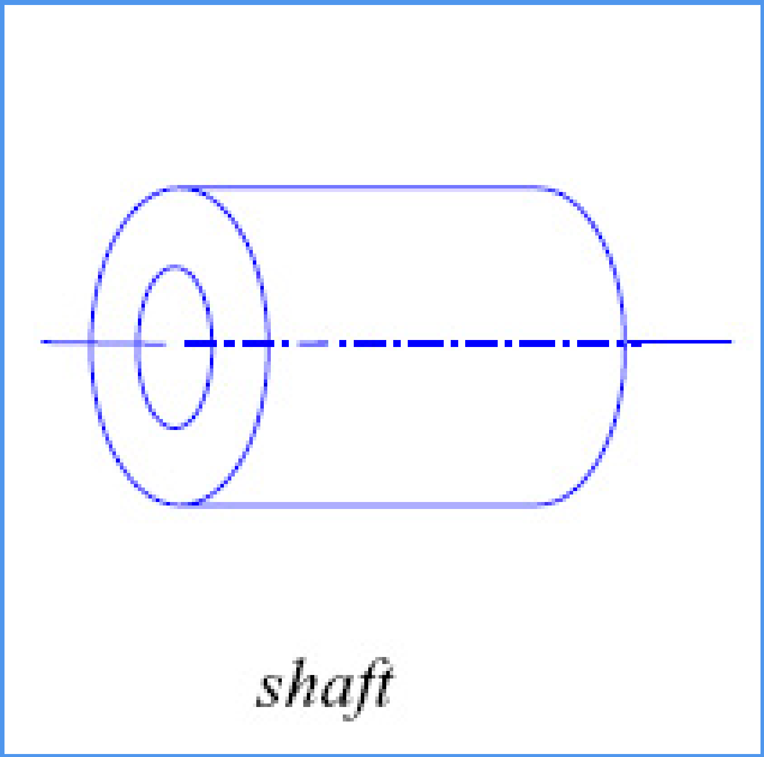
von Mises stress:

$$\sigma_e = \sigma_{VM} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}.$$

Stresses are evaluated at selected points (including nodes) on each element. Averaging (around a node, for example) may be employed to smooth the field.

Solids of Revolution (Axisymmetric Solids):

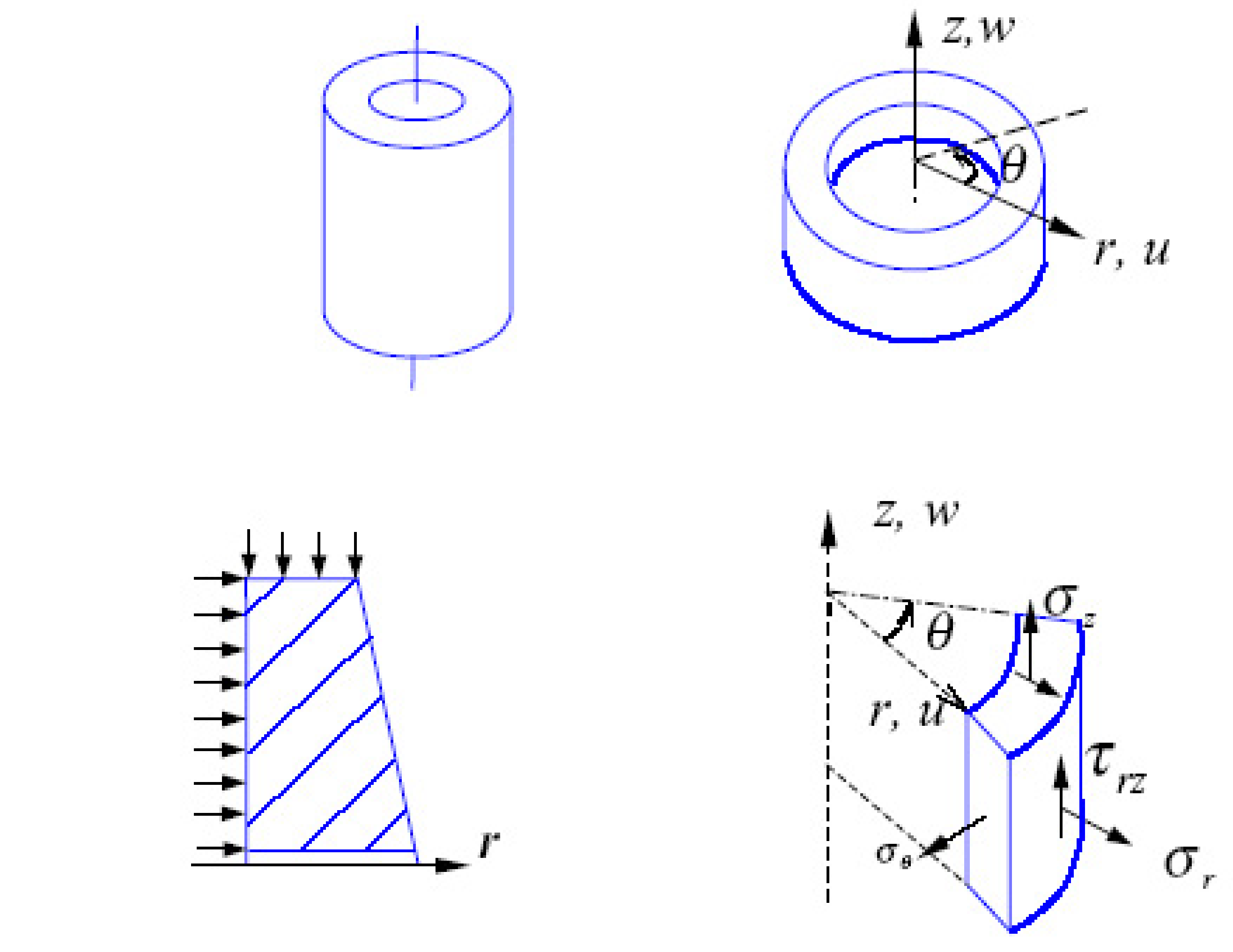




Solids of Revolution (Axisymmetric Solids):

Apply cylindrical coordinates:

$$(x, y, z) \Rightarrow (r, \theta, z)$$



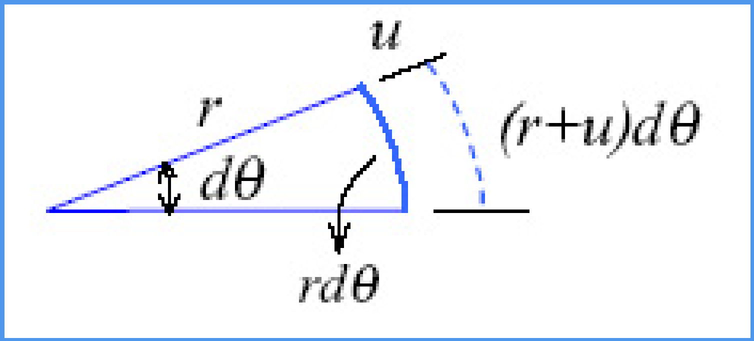
Solids of Revolution (Axisymmetric Solids):

Displacement field:

$u = u(r, z), w = w(r, z)$ (No v - circumferential component)

Strain:

$$\varepsilon_r = \frac{\partial u}{\partial r}, \qquad \varepsilon_\theta = \frac{u}{r}, \qquad \varepsilon_z = \frac{\partial w}{\partial z},$$
$$\gamma_{rz} = \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}, \qquad (\gamma_{r\theta} = \gamma_{z\theta} = 0) \qquad (50)$$

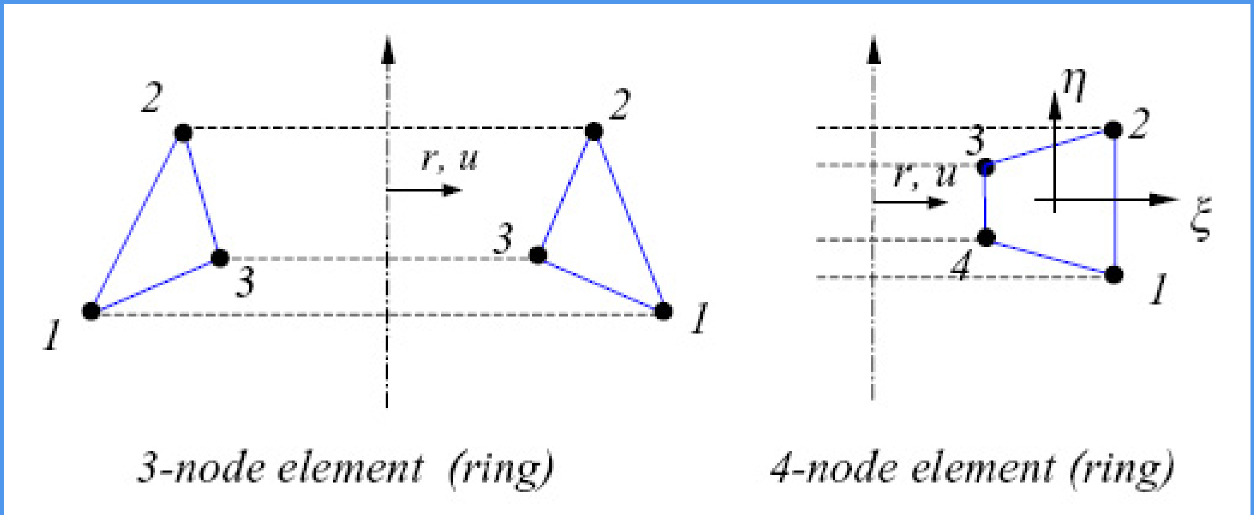


Solids of Revolution (Axisymmetric Solids):

Stresses:

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{rz} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ \gamma_{rz} \end{Bmatrix} \tag{51}$$

Axisymmetric Elements:



$$\mathbf{k} = \int_V \mathbf{B}^T \mathbf{E}\,\mathbf{B}\, r \mathrm{d}r\, \mathrm{d}\theta\, \mathbf{d}z \tag{52}$$

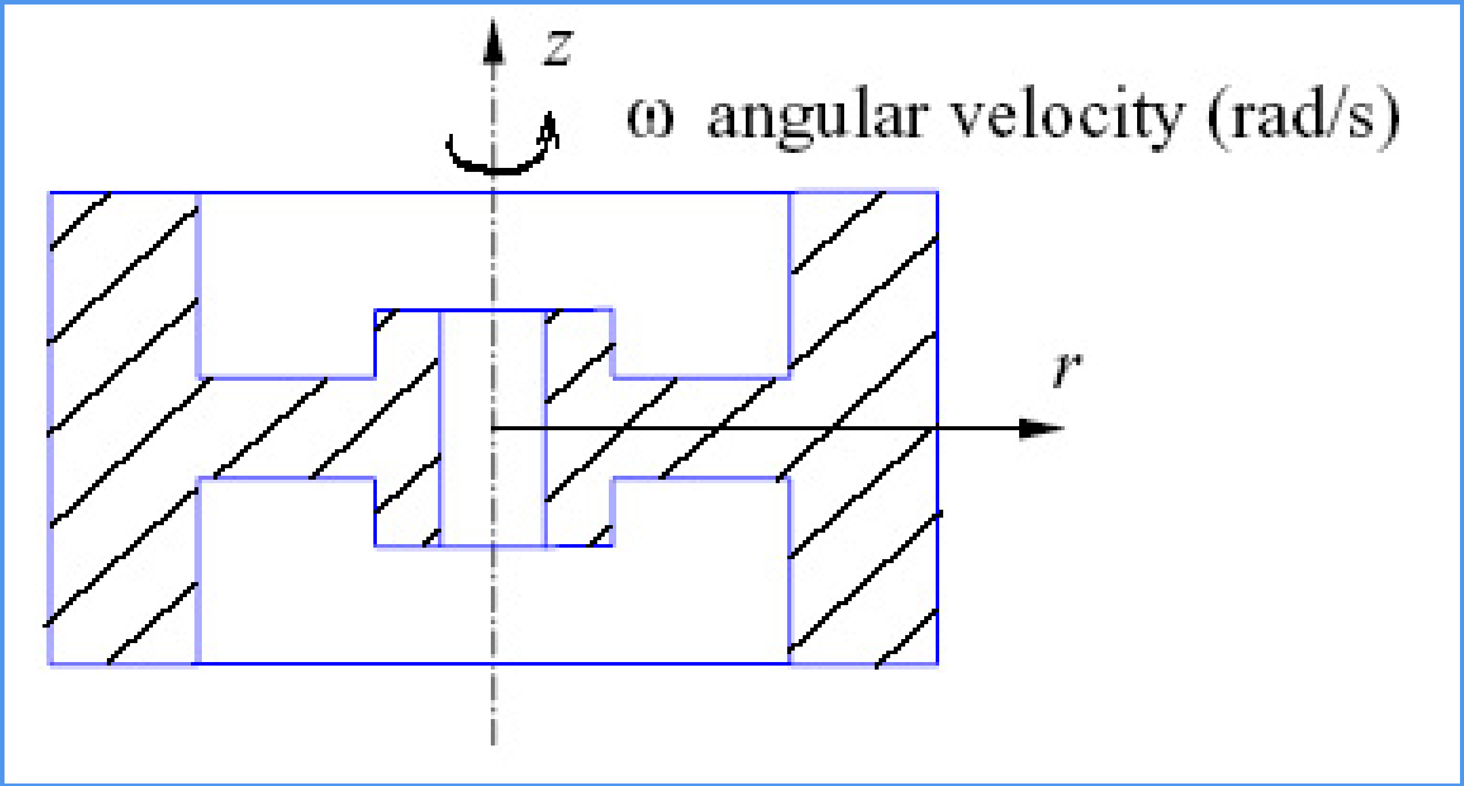
Solids of Revolution (Axisymmetric Solids):

or

$$\mathbf{k} = \int_0^{2\pi} \int_{-1}^{1} \int_{-1}^{1} \mathbf{B}^T \mathbf{E}\, \mathbf{B}\, r(\det \mathbf{J}) \mathrm{d}\xi \, \mathrm{d}\eta \, \mathrm{d}\theta = 2\pi \int_{-1}^{1} \int_{-1}^{1} \mathbf{B}^T \mathbf{E}\, \mathbf{B}\, r(\det \mathbf{J}) \mathrm{d}\xi \, \mathrm{d}\eta \quad (53)$$

Solids of Revolution (Axisymmetric Solids): - Application

- **Rotating Flywheel:**



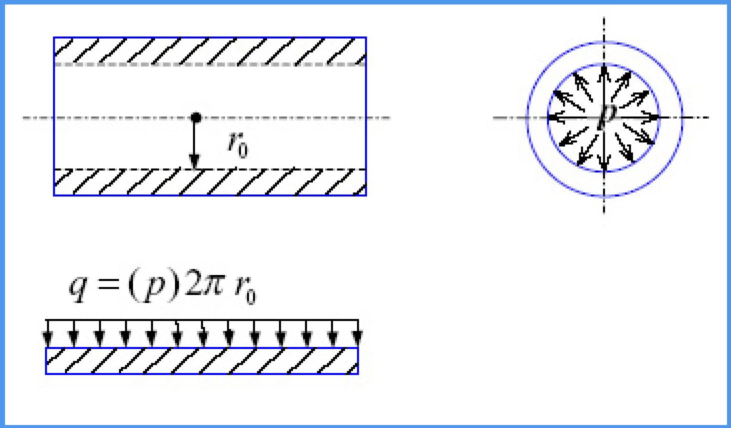
Body forces:

$f_r = \rho\, r\, \omega^2$ (equivalent radial centrifugal / inertial force)

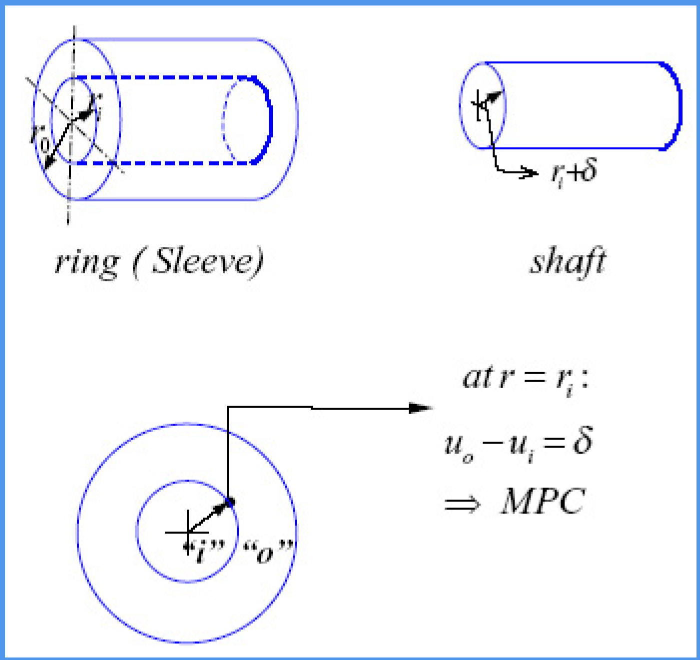
$f_z = -\rho\, g$ (gravitational force)

Solids of Revolution (Axisymmetric Solids): - Application

- **Cylinder Subject to Internal Pressure:**

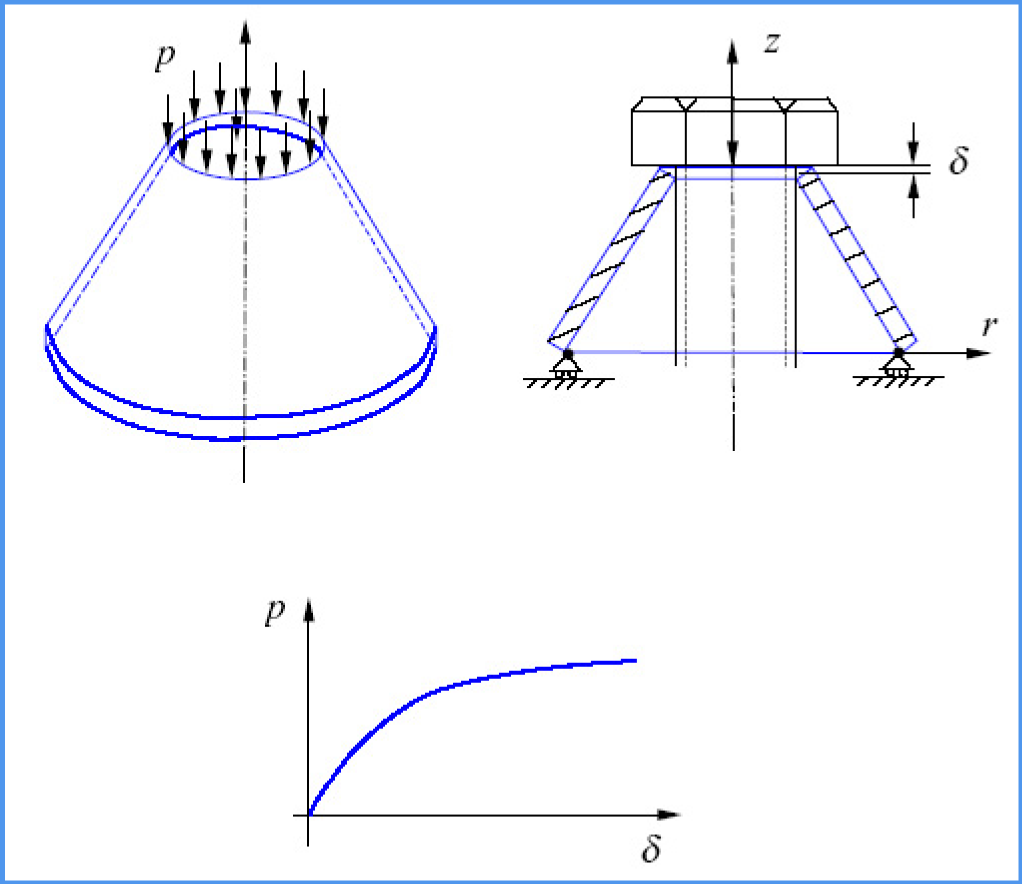


- **Press Fit:**



Solids of Revolution (Axisymmetric Solids): - Application

- **Belleville (Conical) Spring:**



This is a geometrically nonlinear (large deformation) problem and iteration method (incremental approach) needs to be employed.

Učebný text bol pripravený použitím
LaTeXu a balíka PPower4.